Plane Wave Geometry and Quantum Physics

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Abstract. I explain how the Lewis–Riesenfeld exact treatment of the time-dependent quantum harmonic oscillator can be understood in terms of the geodesics and isometries of a plane wave metric, and I show how a curious equivalence between two classes of Yang-Mills actions can be traced back to the transformation relating plane waves in Rosen and Brinkmann coordinates.

Contents

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1. Introduction

The characteristic interplay of geometry and gauge theory in string theory has led to many new and exciting developments in recent years, in particular to progress in the understanding of certain strongly coupled quantum field theories. However, since string theorists were (regrettably) absent from the list of speakers at this conference dedicated to recent developments in quantum field theory, I decided to talk about a subject on the interface of geometry and quantum physics that is only loosely inspired by, and not strictly dependent upon, string theory.[1]

Thus, as an embryonic example of the interplay between geometry and quantum physics in string theory (an example that requires neither knowledge nor appreciation of string theory, but also does not do justice to the depth and richness of these ideas in the string theory context), I will explain the relation between some geometric properties of plane wave space-time metrics on the one hand and some corresponding statements about quantum (gauge) theories on the other.

In section 2, I briefly review some of the basic and entertaining features of the geometry of plane wave metrics. In particular, I emphasise the ubiquitous and multifaceted role of the time-dependent harmonic oscillator in this context, which appears in the geodesic equations, in the description of the Heisenberg isometry algebra of plane wave metrics, and in the coordinate transformation beween the two standard (Rosen and Brinkmann) coordinate systems for these metrics.

The first application I will discuss is then naturally to the quantum theory of time-dependent harmonic oscilators (section 3). In general one can quantise these systems exactly using the powerful Lewis–Riesenfeld method of invariants. Embedding the problem of a time-dependent harmonic oscillator into the plane wave setting equips it with a rich geometric structure, and links the dynamics of the harmonic oscillator to the conserved charges associated with the isometries. I will show that this provides a natural geometric explanation of the entire Lewis–Riesenfeld procedure.

As a second application, I will discuss a curious equivalence between two a priori apparently quite different classes of Yang-Mills theories (section 4). Once again, it is the plane wave perspective which provides an explanation for this. Namely, I will show that this equivalence can be traced back to the coordinate transformation relating plane waves in Rosen and Brinkmannn coordinates, and I add a few comments on what is the string theory context for these particular Yang-Mills actions.

Section 2 is extracted (and adapted to present purposes) from my unpublished lecture notes on plane waves and Penrose limits [1]. The material in section 3 is based on [2], and section 4 is based on currently unpublished material that will appear in [3].

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[1] In retrospect this was a wise choice, because of the hostile attitude towards string theory at this, in all other respects very charming and enjoyable, meeting, expressed in particular by some of the members of the senior pontificating classes.

2. A brief introduction to the geometry of plane wave metrics

2.1. Plane waves in Rosen and Brinkmann coordinates: heuristics

Usually gravitational plane wave solutions of general relativity are discussed in the context of the linearised theory. There one makes the ansatz that the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where $h_{\mu\nu}$ is treated as a small perturbation of the Minkowski background metric $\eta_{\mu\nu}$. To linear order in $h_{\mu\nu}$, the Einstein equations (necessarily) reduce to a wave equation. One finds that gravitational waves are transversally polarised. For example, a wave travelling in the (t, z)-direction distorts the metric only in the transverse directions, and a typical solution of the linearised Einstein equations is

$$ds^2 = -dt^2 + dz^2 + (\delta_{ij} + h_{ij}(z-t))dy^i dy^j \quad . \tag{2}$$

Note that in terms of lightcone coordinates $U = z - t$, $V = (z + t)/2$ this can be written as

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))dy^i dy^j \quad . \tag{3}$$

We will now simply define a plane wave metric in general relativity to be a metric of the above form, dropping the assumption that h_{ij} be "small",

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j \quad . \tag{4}$$

We will say that this is a plane wave metric in *Rosen coordinates.* This is not the coordinate system in which plane waves are usually discussed, among other reasons because typically in Rosen coordinates the metric exhibits spurious coordinate singularities.

Another way of introducing (or motivating the definition of) these plane wave metrics is to start with the $D = d + 2$ dimensional Minkowski metric written in lightcone coordinates,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = 2dudv + \delta_{ab}dx^a dx^b \quad , \tag{5}$$

with $a = 1, \ldots, d$. To this metric one adds a term corresponding to a perturbation travelling at the speed of light in the v-direction,

$$ds^2 = 2dudv + A(u, x^a)(du)^2 + \delta_{ab}dx^a dx^b \quad , \tag{6}$$

and requires that the effect of this term is to exert a linear (harmonic) force on test particles, leading to

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b(du)^2 + \delta_{ab}dx^a dx^b \quad . \tag{7}$$

This is the metric of a plane wave in *Brinkmann coordinates.* We will see below that the two classes of metrics described by (4) and (7) are indeed equivalent. Every metric of the form (4) can be brought to the form (7), and conversely every metric of the type (7) can be written, in more than one way, as in (4).

These exact gravitational plane wave solutions have been discussed in the context of four-dimensional general relativity for a long time (see e.g. [4] and [5]), even though they are not (and were never meant to be) phenomenologically realistic models of gravitational plane waves. The reason for this is that in the far-field gravitational waves are so weak that the linearised Einstein equations and their solutions are adequate to describe the physics, whereas the near-field strong gravitational effects responsible for the production of gravitational waves, for which the linearised equations are indeed insufficient, correspond to much more complicated solutions of the Einstein equations (describing e.g. two very massive stars orbiting around their common center of mass).

Rather, as the in some sense simplest non-trivial genuinely Lorentzian metrics, and as exact solutions of the full non-linear Einstein equations (see section 2.2), these plane wave metrics have always been extremely useful as a theoretical play-ground. It has also long been recognised that gravitational wave metrics provide potentially exact and exactly solvable string theory backgrounds, and this led to a certain amount of activity in this field in the early 1990s (see e.g. [6] for a review). More recently, the observations in [7, 8, 9] have led to a renewed surge in interest in the subject in the string theory community, in particular in connection with the remarkable BMN correspondence [10].

In the following, however, we will just be interested in certain aspects of the geometry of plane waves *per se*, and the role they play in elucidating certain properties of much simpler physical systems. The most basic aspects of the geometry of a space-time metric are revealed by studying its curvature, geodesics, and isometries. This is actually all we need, and we will now address these issues in turn.

2.2. Curvature of plane waves

While not strictly neded for the applications in sections 3 and 4, this brief discussion of the curvature of plane waves provides some useful insight into the geometry and physics of plane waves and the nature of Brinkmann coordinates.

Since the plane wave metric in Brinkmann coordinates (7) is so simple, it is straightforward to see that the only non-vanishing components of its Riemann curvature tensor are

$$R_{uaub} = -A_{ab} \quad . \tag{8}$$

In particular, therefore, there is only one non-trivial component of the Ricci tensor,

$$R_{uu} = -\delta^{ab} A_{ab} \quad , \tag{9}$$

and the Ricci scalar is zero.

Thus the metric is flat iff $A_{ab} = 0$. Moreover, we see that in Brinkmann coordinates the vacuum Einstein equations reduce to a simple algebraic condition on A_{ab} (regardless of its u-dependence), namely that it be traceless. The number of degrees of freedom of this traceless matrix $A_{ab}(u)$ correspond precisely to those of a transverse traceless symmetric tensor (a.k.a. a graviton). In four dimensions, the general vacuum plane wave solution thus has the form

$$ds^2 = 2dudv + [A(u)(x^2 - y^2) + 2B(u)xy]du^2 + dx^2 + dy^2 \tag{10}$$

for arbitrary fuctions $A(u)$ and $B(u)$. This reflects the two polarisation states or degrees of freedom of a four-dimensional graviton. This family of exact solutions to the full non-linear Einstein equations would deserve to have text-book status but does not, to the best of my knowledge, appear in any of the standard introductory texts on general relativity.

2.3. Geodesics, lightcone gauge and harmonic oscillators

We now take a look at geodesics of a plane wave metric in Brinkmann coordinates, i.e. the solutions $x^\mu(\tau)$ to the geodesic equations

$$\ddot{x}^\mu(\tau) + \Gamma^\mu_{\ \nu\lambda}(x(\tau))\dot{x}^\nu(\tau)\dot{x}^\lambda(\tau) = 0 \quad , \tag{11}$$

where an overdot denotes a derivative with respect to the affine parameter τ. Rather than determining the geodesic equations by first calculating all the non-zero Christoffel symbols, we make use of the fact that the geodesic equations can be obtained more efficiently, and in a way that allows us to directly make use of the symmetries of the problem, as the Euler-Lagrange equations of the Lagrangian

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \\ &= \dot{u}\dot{v} + \tfrac{1}{2}A_{ab}(u)x^a x^b \dot{u}^2 + \tfrac{1}{2}\dot{\vec{x}}^2 \quad , \end{aligned} \tag{12}$$

supplemented by the constraint $2\mathcal{L} = \epsilon$, where $\epsilon = 0$ ($\epsilon = -1$) for massless (massive) particles. Since nothing depends on v, the *lightcone momentum*

$$p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \dot{u} \tag{13}$$

is conserved. For $p_v = 0$, the particle obviously does not feel the curvature term A_{ab}, and the geodesics are straight lines. When $p_v \neq 0$, one has $u = p_v\tau + u_0$, and by an affine transformation of τ one can always choose the *lightcone gauge*

$$u = \tau \quad . \tag{14}$$

Then the geodesic equations for the transverse coordinates are the Euler-Lagrange equations

$$\ddot{x}^a(\tau) = A_{ab}(\tau)x^b(\tau) \quad . \tag{15}$$

These are the equation of motion of a non-relativistic *harmonic oscillator* with (possibly time-dependent) frequency matrix $\omega^2_{ab}(\tau) = -A_{ab}(\tau)$. The constraint $2\mathcal{L} = \epsilon$, or

$$2\dot{v}(\tau) + A_{ab}(\tau)x^a(\tau)x^b(\tau) + \delta_{ab}\dot{x}^a(\tau)\dot{x}^b(\tau) = \epsilon, \tag{16}$$

is then a first integral of the equation of motion for the remaining coordinate $v(\tau)$, and is readily integrated to give $v(\tau)$ in terms of the solutions to the harmonic oscillator equation for $x^a(\tau)$.

By definition the lightcone Hamiltonian is (minus!) the momentum p_u conjugate to u (in the lightcone gauge $u = \tau$),

$$H_{lc} = -p_u \quad . \tag{17}$$

Using

$$p_u = g_{u\mu}\dot{x}^\mu = \dot{v} + A_{ab}(\tau)x^a x^b \tag{18}$$

and the constraint, one finds that the lightcone Hamiltonian is just (for $\epsilon \neq 0$ up to an irrelevant constant) the Hamiltonian of the above harmonic oscillator,

$$H_{lc} = \tfrac{1}{2}(\delta_{ab}\dot{x}^a\dot{x}^b - A_{ab}(\tau)x^a x^b) - \tfrac{1}{2}\epsilon \equiv H_{ho} - \tfrac{1}{2}\epsilon \ . \tag{19}$$

Note also that, in the lightcone gauge, the complete relativistic particle Lagrangian

$$\mathcal{L} = \dot{v} + \tfrac{1}{2}A_{ab}(\tau)x^a x^b + \tfrac{1}{2}\dot{\vec{x}}^2 = \mathcal{L}_{ho} + \dot{v} \tag{20}$$

differs from the harmonic oscillator Lagrangian only by a total time-derivative.

In summary, we note that in the lightcone gauge the equations of motion of a *relativistic particle* in the plane wave metric reduce to those of a *non-relativistic harmonic oscillator*. This harmonic oscillator equation plays a central role in the following and will reappear several times below in different contexts, e.g. when discussing the transformation from Rosen to Brinkmann coordinates, or when analysing the isometries of a plane wave metric.

2.4. From Rosen to Brinkmann coordinates (and back)

I will now describe the relation between the plane wave metric in Brinkmann coordinates,

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \ , \tag{21}$$

and in Rosen coordinates,

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j \ . \tag{22}$$

It is clear that, in order to transform the non-flat transverse metric in Rosen coordinates to the flat transverse metric in Brinkmann coordinates, one should change variables as

$$x^a = E^a_{\ i} y^i \ , \tag{23}$$

where $E^a_{\ i}$ is a vielbein for g_{ij} in the sense that

$$g_{ij} = E^a_{\ i} E^b_{\ j}\delta_{ab} \ . \tag{24}$$

Plugging this into the metric, one sees that this has the desired effect provided that E satisfies the symmetry condition

$$\dot{E}_{ai}E^i_{\ b} = \dot{E}_{bi}E^i_{\ a} \tag{25}$$

(such an E can always be found), and provided that one accompanies this by a shift in V. The upshot of this is that the change of variables

$$\begin{aligned} U &= u \\ V &= v + \tfrac{1}{2}\dot{E}_{ai}E^i_{\ b}x^a x^b \\ y^i &= E^i_{\ a}x^a \ , \end{aligned} \tag{26}$$

transforms the Rosen coordinate metric (22) into the Brinkmann form (21), with A_{ab} given by [2]

$$A_{ab} = \ddot{E}_{ai}E^i_{\ b} \ . \tag{27}$$

This can also be written as the *harmonic oscillator equation* (again!)

$$\ddot{E}_{ai} = A_{ab}E_{bi} \tag{28}$$

we had already encountered in the context of the geodesic equation.

In practice, once one knows that Rosen and Brinkmann coordinates are indeed just two distinct ways of describing the same class of metrics, one does not need to perform explicitly the coordinate transformation mapping one to the other. All one is interested in is the relation between $g_{ij}(U)$ and $A_{ab}(u)$, which is just the relation (8)

$$A_{ab} = -E^i_{\ a} E^j_{\ b} R_{UiUj} = -R_{uaub} \tag{29}$$

between the curvature tensor in Rosen and Brinkmann coordinates.

There is a lot of nice geometry lurking behind the transformation from Rosen to Brinkmann coordinates. For example, the symmetry condition (25) says that $E^i_{\ a}$ is a parallel transported co-frame along the null geodesic congruence ($U = \tau, V, y^i = const.$) [11, 12], and the coordinate transformation itself can be interpreted as passing from Rosen coordinates to inertial Fermi coordinates adapted to the null geodesic ($U = \tau, V = 0, y^i = 0$) [13].

For a different perspective, and a prescription for how to go back from Brinkmann to Rosen coordinates, note that the index i on E_{ai} in (28) can be thought of as labelling d out of the $2d$ linearly independent solutions of the oscillator equation. The symmetry condition (25) can equivalently be written as

$$\dot{E}_{ai} E^i_{\ b} = \dot{E}_{bi} E^i_{\ a} \quad \Leftrightarrow \quad \dot{E}_{ai} E^a_{\ k} = \dot{E}_{ak} E^a_{\ i} \ \ , \tag{30}$$

and can now be interpreted as the condition that the *Wronskian* of the i'th and k'th solution

$$W(E_i, E_k) := \dot{E}_{ak} E^a_{\ i} - \dot{E}_{ai} E^a_{\ k} \tag{31}$$

is zero. Thus, given the metric in Brinkmann coordinates, one can construct the metric in Rosen coordinates by solving the oscillator (geodesic) equation, choosing a maximally commuting set of solutions to construct E_{ai}, and then determining g_{ij} algebraically from the E_{ai} from (24).

2.5. The Heisenberg isometry algebra of a generic plane wave

We now study the isometries of a generic plane wave metric. In Brinkmann coordinates, because of the explicit dependence of the metric on u and the transverse coordinates, only one isometry is manifest, namely that generated by the parallel (covariantly constant) and hence in particular Killing null vector $Z = \partial_v$. In Rosen coordinates, the metric depends neither on V nor on the transverse coordinates y^k, and one sees that in addition to $Z = \partial_V$ there are at least d more Killing vectors, namely the ∂_{y^k}. Together these form an Abelian translation algebra acting transitively on the null hypersurfaces of constant U.

However, this is not the whole story. Indeed, one particularly interesting and peculiar feature of plane wave space-times is the fact that they generically possess a *solvable* (rather than semi-simple) isometry algebra, namely a Heisenberg algebra, only part of which we have already seen above.

All Killing vectors X can be found in a systematic way by solving the Killing equations

$$L_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \ \ . \tag{32}$$

I will not do this here but simply present the results of this analysis in Brinkmann coordinates (see [2] for details). The upshot is that a generic $(2+d)$-dimensional plane wave metric has a $(2d+1)$-dimensional isometry algebra generated by the Killing vector $Z=\partial_v$ and the $2d$ Killing vectors

$$X(f_{(K)}) \equiv X_{(K)} = f_{(K)a}\partial_a - \dot{f}_{(K)a}x^a\partial_v \quad . \tag{33}$$

Here the $f_{(K)a}$, $K=1,\dots,2d$ are the $2d$ linearly independent solutions of the *harmonic oscillator equation* (yet again!)

$$\ddot{f}_a(u) = A_{ab}(u)f_b(u) \quad . \tag{34}$$

These Killing vectors satisfy the algebra

$$[X_{(J)},X_{(K)}] = -W(f_{(J)},f_{(K)})Z \tag{35}$$

$$[X_{(J)},Z] = 0 \quad , \tag{36}$$

where the Wronskian $W(f_{(J)},f_{(K)})$ is, exactly as in (31), given by

$$W(f_{(J)},f_{(K)}) = \sum_a (f_{(J)a}\dot{f}_{(K)a} - f_{(K)a}\dot{f}_{(J)a}) \quad . \tag{37}$$

It is of course constant (independent of u) as a consequence of the harmonic oscillator equation. This is already the Heisenberg algebra. To make this more explicit, one can make a convenient choice of basis for the solutions $f_{(J)}$ by splitting the $f_{(J)}$ into two sets of solutions

$$\{f_{(J)}\} \rightarrow \{q_{(a)},p_{(a)}\} \tag{38}$$

characterised by the initial conditions

$$\begin{aligned} q_{(a)b}(u_0) = \delta_{ab} \quad & \dot{q}_{(a)b}(u_0) = 0 \\ p_{(a)b}(u_0) = 0 \quad & \dot{p}_{(a)b}(u_0) = \delta_{ab} \quad . \end{aligned} \tag{39}$$

Since the Wronskian of these functions is independent of u, it can be determined by evaluating it at $u=u_0$. Then one can immediately read off that

$$\begin{aligned} & W(q_{(a)},q_{(b)}) = W(p_{(a)},p_{(b)}) = 0 \\ & W(q_{(a)},p_{(b)}) = \delta_{ab} \quad . \end{aligned} \tag{40}$$

Therefore the corresponding Killing vectors

$$Q_{(a)} = X(q_{(a)}) \quad , \qquad P_{(a)} = X(p_{(a)}) \tag{41}$$

and Z satisfy the canonically normalised Heisenberg algebra

$$\begin{aligned} & [Q_{(a)},Z] = [P_{(a)},Z] = 0 \\ & [Q_{(a)},Q_{(b)}] = [P_{(a)},P_{(b)}] = 0 \\ & [Q_{(a)},P_{(b)}] = -\delta_{ab}Z \quad . \end{aligned} \tag{42}$$

As we had noted before, in Rosen coordinates, the $(d+1)$ translational isometries in the V and y^k directions, generated by the Killing vectors $Z=\partial_V$ and

$Q_{(k)} = \partial_{y^k}$, are manifest. One can check that the "missing" d Killing vectors $P_{(k)}$ are given by

$$P_{(k)} = -y^k \partial_V + \int^u du' \; g^{km}(u') \partial_{y^m} \quad . \tag{43}$$

It is straightforward to verify that together they also generate the Heisenberg algebra (42).

These considerations also provide yet another perspective on the transformation from Brinkmann to Rosen coordinates, and the vanishing Wronskian condition discussed at the end of section 2.4. Indeed, passing from Brinkmann to Rosen coordinates can be interpreted as passing to coordinates in which half of the translational Heisenberg algebra symmetries are manifest. This is achieved by choosing the (transverse) coordinate lines to be the integral curves of these Killing vectors. This is of course only possible if these Killing vectors commute, i.e. the Wronskian of the corresponding solutions of the harmonic oscillator equation is zero, and results in a metric which is independent of the transverse coordinates, namely the plane wave metric in Rosen coordinates.

2.6. Geodesics, isometries, and conserved charges

We can now combine the results of the previous sections to determine the conserved charges carried by particles moving geodesically in the plane wave geometry. In general, given any Killing vector X, there is a corresponding conserved charge $C(X)$,

$$C(X) = g_{\mu\nu} X^\mu \dot{x}^\mu \quad . \tag{44}$$

That $C(X)$ is indeed constant along the trajectory of the geodesic $x^\mu(\tau)$ can easily be verified by using the geodesic and Killing equations.

The conserved charge corresponding to the Killing vector $Z = \partial_v$, the central element of the Heisenberg algebra, is, none too surprisingly, nothing other than the conserved lightcone momentum p_v (13) of section 2.3,

$$C(Z) = g_{v\mu} \dot{x}^\mu = \dot{u} = p_v \quad . \tag{45}$$

In addition to Z, for any solution f of the harmonic oscilator equation we have a Killing vector $X(f)$ (33),

$$X(f) = f_a \partial_a - \dot{f}_a x^a \partial_v \quad . \tag{46}$$

The associated conserved charge is

$$C(X(f)) = f_a p^a - \dot{f}_a x^a \quad . \tag{47}$$

(here we have used the, now more appropriate, phase space notation $p^a = \dot{x}^a$). This is rather trivially conserved (constant), since both f_a and x^a are solutions of the same ubiquitous harmonic oscillator equation and $C(X(f))$ is nothing other than their constant Wronskian,

$$C(X(f)) = W(f, x) \quad . \tag{48}$$

Thus these somewhat tautological conserved charges are not helpful in integrating the geodesic or harmonic oscillator equations. Nevertheless, the very fact that they exist, and that they satisfy a (Poisson bracket) Heisenberg algebra, will turn out to be conceptually important in section 3. We will denote the conserved charges corresponding to the Killing vectors $Q_{(a)}$ and $P_{(a)}$ (41) by

$$C(Q_{(a)}) \equiv \mathcal{Q}_{(a)} \qquad C(P_{(a)}) \equiv \mathcal{P}_{(a)} \quad . \tag{49}$$

The Poisson brackets among the charges $C(X(f))$ can be determined from the canonical Poisson brackets $\{x^a, p^b\} = \delta_{ab}$ to be

$$\{X(f_1), X(f_2)\} = \{f_{1a}p^a - \dot{f}_{1a}x^a, f_{2a}p^a - \dot{f}_{2a}x^a\} = W(f_1, f_2) \tag{50}$$

(note the usual sign flip with respect to the Lie bracket (35) of the corresponding vector fields). In particular, as a consequence of (40) the charges $\mathcal{Q}_{(a)}$ and $\mathcal{P}_{(a)}$ have the canonical Poisson brackets

$$\{\mathcal{Q}_{(a)}, \mathcal{P}_{(b)}\} = \delta_{ab} \quad . \tag{51}$$

Generically, a plane wave metric has just this Heisenberg algebra of isometries which acts transitively on the null hyperplanes $u = const.$, with a simply transitive Abelian subalgebra. However, for special choices of $A_{ab}(u)$, there may of course be more Killing vectors. These could arise from internal symmetries of A_{ab}, giving more Killing vectors (and corresponding conserved angular momenta) in the transverse directions, as for an isotropic harmonic oscillator.

Of more interest is the fact that for particular $A_{ab}(u)$ there may be Killing vectors with a ∂_u-component. The existence of such a Killing vector renders the plane wave homogeneous (away form the fixed points of this extra Killing vector). These homogeneous plane waves have been completely classified in [2]. The simplest examples, and the only ones that we will consider here, are plane waves with a u-independent profile A_{ab},

$$ds^2 = 2dudv + A_{ab}x^a x^b du^2 + d\vec{x}^2 \quad , \tag{52}$$

which obviously, since now nothing depends on u, have the extra Killing vector $X = \partial_u$.

The existence of the additional Killing vector $X = \partial_u$ extends the Heisenberg algebra to the harmonic oscillator algebra, with X playing the role of the number operator or harmonic oscillator Hamiltonian. Indeed, X and $Z = \partial_v$ obviously commute, and the commutator of X with one of the Killing vectors $X(f)$ is

$$[X, X(f)] = X(\dot{f}) \quad . \tag{53}$$

Note that this is consistent, i.e. the right-hand-side is again a Killing vector, because when A_{ab} is constant and f satisfies the harmonic oscillator equation then so does its u-derivative $\dot{f}$. In terms of the basis (41) we have

$$\begin{aligned} [X, Q_{(a)}] &= P_{(a)} \\ [X, P_{(a)}] &= A_{ab}Q_{(b)} \quad , \end{aligned} \tag{54}$$

which is the harmonic oscillator algebra.

Another way of understanding the relation between $X = \partial_u$ and the harmonic oscillator Hamiltonian is to look at the conserved charge associated with $X = \partial_u$,

$$C(\partial_u) = g_{u\mu}\dot{x}^\mu = p_u \quad , \tag{55}$$

which we had already identified (up to a constant for non-null geodesics) as minus the harmonic oscillator Hamiltonian in section 2.3. This is of course indeed a conserved charge iff the Hamiltonian is time-independent, i.e. iff A_{ab} is constant.

2.7. Synopsis

In the above I have reviewed, in somewhat more detail than strictly necessary for the following, some of the interesting and entertaining aspects of the geometry of plane wave metrics. The only things that we will actually directly make use of below are, in section 3,

- the Heisenberg isometry algebra (section 2.5)
- and the existence of the corresponding conserved charges (section 2.6),

and, in section 4,

- the lightcone gauge geodesic Lagrangian (section 2.3)
- and the transformation from Rosen to Brinkmann coordinates (section 2.4).

3. The Lewis–Riesenfeld theory of the time-dependent quantum oscillator

3.1. Description of the problem

We will now discuss the quantum theory of a time-dependent harmonic oscillator (for simplicity in $d = 1$ dimension, but the discussion generalises in an obvious way to $d > 1$), with Hamiltonian

$$H_{ho}(t) = \tfrac{1}{2}(p^2 + \omega(t)^2 x^2) \quad . \tag{56}$$

The aim is to find the solutions of the time-dependent Schrödinger equation (in units with $\hbar = 1$)

$$i\partial_t|\psi(t)\rangle = \hat{H}_{ho}(t)|\psi(t)\rangle \quad . \tag{57}$$

Standard textbook treatments of this problem employ the following strategy:

- When the Hamiltonian is time-independent, then the standard procedure is of course to reduce this problem to that of finding the stationary eigenstates $|\psi_n\rangle$ of $\hat{H}_{ho}$,

  $$\hat{H}_{ho}|\psi_n\rangle = E_n|\psi_n\rangle \quad , \tag{58}$$

  with $E_n = \omega(n + \frac{1}{2})$ etc., in terms of which the general solution to the time-dependent Schrödinger equation can then be written as

  $$|\psi(t)\rangle = \sum_n c_n \mathrm{e}^{-iE_n t}|\psi_n\rangle \quad , \tag{59}$$

  where the c_n are constants.

- When the Hamiltonian is time-dependent, on the other hand, then in principle the solution is given by the time-ordered exponential of $\hat{H}_{ho}(t)$,

$$|\psi(t)\rangle = \left(\mathcal{T}\mathrm{e}^{-i\int_{t_0}^{t} dt'\ \hat{H}_{ho}(t')}\right)|\psi(t_0)\rangle \ \ , \tag{60}$$

but in practice this cannot be evaluated to get an exact solution. One thus needs to then invoke some kind of adiabatic approximation to perturbatively determine the solution (and then calculate transition and decay rates etc.).

What Lewis and Riesenfeld observed [14] is that, even in the time-dependent case, there is a procedure analogous to the one used in the time-independent case which allows one to explicitly find the exact solutions of the time-dependent Schödinger equation.

3.2. Outline of the Lewis–Riesenfeld procedure

The idea of [14] is to base the construction of the solutions of the Schrödinger equation not on the stationary eigenstates of the Hamiltonian (which does not make sense when the Hamiltonian depends explicitly on time) but on the eigenstates of another operator $\hat{I}$ which is an *invariant* of the system. This means that

$$\hat{I}(t,\hat{x},\hat{p}) \equiv \hat{I}(t) \tag{61}$$

is a (typically explicitly time-dependent) operator satisfying

$$i\tfrac{d}{dt}\hat{I}(t) \equiv i\partial_t\hat{I}(t) + [\hat{I}(t),\hat{H}_{ho}(t)] = 0 \tag{62}$$

(when $\hat{H}_{ho}$ is time-independent, then one can of course just take $\hat{I} = \hat{H}_{ho}$). The Lewis–Riesenfeld procedure now consists of two parts:

1. The first is to show how one can construct all the solutions of the time-dependent Schrödinger equation for $\hat{H}_{ho}(t)$ from the spectrum and eigenstates of the invariant $\hat{I}(t)$.
2. The second is an algorithm which provides an invariant for any time-dependent harmonic oscillator, and which moreover has the feature that $\hat{I}(t)$ itself has the form of a *time-independent* harmonic oscillator (so that it is straightforward to determine the spectrum and eigenstates of $\hat{I}(t)$).

Issue (1) can be established by straightforward and relatively standard quantum mechanical manipulations. I will briefly recall these below but have nothing new to add to that part of the discussion. Issue (2), on the other hand, is usually established by a direct but rather brute-force calculation which does not appear to provide any conceptual insight into why invariants with the desired properties exist. I will show in section 3.3 that this conceptual insight is obtained by embedding the time-dependent harmonic oscillator into the plane wave setting.

To address (1), let us assume that an invariant $\hat{I}(t)$ satisfying (62) exists and that it is hermitian. We choose a complete set of eigenstates, labelled by the real eigenvalues λ of $\hat{I}(t)$,

$$\hat{I}(t)|\lambda\rangle = \lambda|\lambda\rangle \ \ . \tag{63}$$

It follows from (62) that the eigenvalues λ are time-independent, and that

$$\langle\lambda'|i\partial_t - \hat{H}_{ho}(t)|\lambda\rangle = 0 \tag{64}$$

for all $\lambda \neq \lambda'$. We would like this equation to be true also for the diagonal elements, in which case we would have already found the solutions of the time-dependent Schrödinger equation for $\hat{H}_{ho}(t)$. To accomplish this, we slightly modify the eigenfunctions by multiplying them by a time-dependent phase,

$$|\lambda\rangle \to \mathrm{e}^{i\alpha_\lambda(t)}|\lambda\rangle \ . \tag{65}$$

It can be seen immediately that this phase factor does not change the off-diagonal matrix elements of $i\partial_t - \hat{H}_{ho}(t)$ (since the eigenstates are orthogonal). Requiring the validity of (64) also for $\lambda = \lambda'$ then leads to a first-order differential equation for $\alpha_\lambda(t)$,

$$\tfrac{d}{dt}\alpha_\lambda(t) = \langle\lambda|i\partial_t - \hat{H}_{ho}(t)|\lambda\rangle. \tag{66}$$

Solving this equation, the general solution to the time-dependent Schrödinger equation for $\hat{H}_{ho}(t)$ is, similarly to (59),

$$|\psi(t)\rangle = \sum_\lambda c_\lambda \mathrm{e}^{i\alpha_\lambda(t)}|\lambda\rangle \ , \tag{67}$$

where the c_λ are constants.

This is an extremely neat way of solving exactly the quantum theory of the time-dependent harmonic oscillator. Its usefulness, however, depends on the ability to construct a suitable invariant $\hat{I}(t)$ which is such that (a) one can explicitly find its spectrum and eigenstates and (b) it is sufficiently closely related to $\hat{H}_{ho}(t)$ so that one can evaluate the diagonal matrix elements of $\hat{H}_{ho}(t)$ in the basis of eigenstates $|\lambda\rangle$ of the invariant $\hat{I}(t)$ (in order to determine the phases $\alpha_\lambda(t)$).

In a nutshell, this is achieved in [14] as follows (see also [2] for a detailed account with further comments on the procedure). Let $\sigma(t)$ be any solution to the non-linear differential equation

$$\ddot{\sigma}(t) + \omega(t)^2\sigma(t) = \sigma(t)^{-3} \ , \tag{68}$$

where $\omega(t)$ is the harmonic oscillator frequency. Then it can be checked by a straightforward but unenlightning calculation that

$$\hat{I}(t) = \tfrac{1}{2}(\hat{x}^2\sigma(t)^{-2} + (\sigma(t)\hat{p} - \dot{\sigma}(t)\hat{x})^2) \tag{69}$$

is an invariant in the sense of (62). As a first sanity check on this construction, note that for ω time-independent one can also choose $\sigma = \omega^{-1/2}$ to be constant, upon which the invariant becomes

$$\hat{I} = \tfrac{1}{2}(\omega\hat{x}^2 + \omega^{-1}\hat{p}^2) = \omega^{-1}\hat{H}_{ho} \ , \tag{70}$$

which is of course the privileged invariant of a time-independent system. In general, in terms of the hermitian conjugate operators

$$\hat{a} = \tfrac{1}{\sqrt{2}}(\hat{x}\sigma^{-1} + i(\sigma\hat{p} - \dot{\sigma}\hat{x})) \qquad \hat{a}^\dagger = \tfrac{1}{\sqrt{2}}(\hat{x}\sigma^{-1} - i(\sigma\hat{p} - \dot{\sigma}\hat{x})) \tag{71}$$

which satisfy the canonical commutation relations $[\hat{a},\hat{a}^\dagger]=1$, $\hat{I}(t)$ has the standard oscillator representation

$$\hat{I}(t) = \hat{a}^\dagger\hat{a} + \tfrac{1}{2} \tag{72}$$

of a time-independent harmonic oscillator, and the original Hamiltonian is a quadratic function of $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{H}_{ho}(t) = c(t)(\hat{a})^2 + c(t)^*(\hat{a}^\dagger)^2 + d(t)(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) \ \ , \tag{73}$$

where e.g. $d(t) = \frac{1}{2}(\omega(t)^2\sigma(t)^2 + \dot{\sigma}(t)^2 + \sigma(t)^{-2})$. This makes it straightforward to evaluate e.g. the diagonal matrix elements of $\hat{H}_{ho}(t)$ in the standard basis of eigenstates of $\hat{I}(t)$.

Finally, the general solution to (68) can be written in terms of any two linearly independent solutions f_1, f_2 to the classical harmonic oscillator equation for $H_{ho}(t)$ (this is (68) with zero on the rhs instead of the non-linear term). Normalising their Wronskian to 1, the general solution $\sigma(t)$ is [14]

$$\sigma = \pm\left[c_1^2 f_1^2 + c_2^2 f_2^2 \pm 2(c_1^2c_2^2 - 1)^{1/2} f_1 f_2\right]^{1/2} \ \ , \tag{74}$$

where c_i are constants subject to the condition that the solution is real, and the signs can be chosen independently.

3.3. Deducing the procedure from the plane wave geometry

While the procedure outlined above provides a concrete (and in practice also very useful) algorithm to solve exactly the quantum theory of a time-dependent harmonic oscillator (and certain other time-dependent systems [14]), it remains somewhat unsatisfactory from a conceptual point of view. In particular, it is not clear from the construction

- why invariants with the desired properties exist in the first place;
- why solutions to the classical equations play a role in the construction of these quantum invariants;
- why one should solve the non-linear equation (68) if, in any case, in the end it all boils down to solutions of the ordinary linear classical harmonic oscillator equation apearing in (74).

Here is where insight is gained by realising the harmonic oscillator equation as the geodesic equation in a plane wave metric. Recall that in section 2.5 we had found a Heisenberg isometry algebra which, in particular, includes the "hidden" symmetries generated by the Killing vector fields (33,46)

$$X(f) = f_a\partial_a - \dot{f}_a x^a \partial_v \ \ , \tag{75}$$

where f is a solution of the classical harmonic oscillator equation, and the corresponding "hidden" conserved charges (47)

$$C(X(f)) = f_a p^a - \dot{f}_a x^a \ \ . \tag{76}$$

In particular, we had obtained the conserved charges $\mathcal{Q}_{(a)}$ and $\mathcal{P}_{(a)}$ (49). These are linear in the phase space variables x^a and p^a, and thus we can unambiguously associate to them quantum operators

$$\mathcal{Q}_{(a)} \to \hat{\mathcal{Q}}_{(a)} \qquad \mathcal{P}_{(a)} \to \hat{\mathcal{P}}_{(a)} \tag{77}$$

which, by construction, are invariants in the sense of (62),

$$\tfrac{d}{dt}\hat{\mathcal{Q}}_{(a)} = \tfrac{d}{dt}\hat{\mathcal{P}}_{(a)} = 0 \ \ , \tag{78}$$

and which satisfy the canonical commutation relations (cf. (51))

$$[\hat{\mathcal{Q}}_{(a)}, \hat{\mathcal{P}}_{(b)}] = i\delta_{ab} \ \ . \tag{79}$$

Note that to "see" these invariants, one has to extend the harmonic oscillator configuration space not just by the time-direction $t = u$, but one also has to add yet another dimension, the null direction v.

The rest is now straightforward. Since $\hat{\mathcal{Q}}_{(a)}$ and $\hat{\mathcal{P}}_{(b)}$ are invariants, also any quadratic operator in these variables (with constant coefficients) is an invariant. In the one-dimensional case ($d = 1$), we can e.g. consider invariants of the form

$$\hat{I}(t) = \tfrac{1}{2M}\hat{\mathcal{P}}^2 + \tfrac{M\Omega^2}{2}\hat{\mathcal{Q}}^2 \ \ , \tag{80}$$

wich we can write in terms of invariant creation and annihilation operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^\dagger$ (constructed in the usual way from $\hat{\mathcal{Q}}$ and $\hat{\mathcal{P}}$) as

$$\hat{I}(t) = \Omega(\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}} + \tfrac{1}{2}) \ \ . \tag{81}$$

Let us now compare this in detail with the results of section 3.2. First of all, to match with the (arbitrary choice of) normalisation of the invariant (72), we choose $\Omega = 1$. Next we can identify what $\sigma(t)^2$ is by identifying it with the coefficient of $\hat{p}^2$ in the expansion of (80) in terms of $\hat{p}$ and $\hat{x}$. The upshot is that $\sigma(t)$ has precisely the form given in (74), with $c_1^2 = 1$ and $c_2^2 = 1/M$. Finally, one sees that the invariant oscillators $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^\dagger$ are related to the oscillators $\hat{a}$ and $\hat{a}^\dagger$ by a unitary transformation which is precisely the unitary transformation that implements the phase transformation (65) on the eigenstates of the invariant.

We have thus come full circle. Starting with the conserved charges associated with the Heisenberg algebra Killing vectors, we have constructed quadratic quantum invariants and have reproduced all the details of the Lewis–Riesenfeld algorithm, including the phase factors $\alpha_\lambda(t)$. Constructing the Fock space in the usual way, one then obtains all the solutions (67) to the time-dependent Schrödinger equation.

4. A curious equivalence between two classes of Yang-Mills actions

4.1. Description of the problem

A prototypical non-Abelian Yang-Mills + scalar action in $n = p+1$ dimensions, obtained e.g. by the dimensional reduction of pure Yang-Mills theory (with standard

Lagrangian $L \sim \mathrm{tr}\, F_{MN}F^{MN}$) in D space-time dimensions down to n dimensions, has the form

$$S_{YM} = \int d^n\sigma\, \mathrm{Tr}\left(-\tfrac{1}{4}g_{YM}^{-2}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} - \tfrac{1}{2}\eta^{\alpha\beta}D_\alpha\phi^a D_\beta\phi^a + \tfrac{1}{4}g_{YM}^2[\phi^a,\phi^b]^2\right) \quad . \tag{82}$$

Here the ϕ^a, $a = 1,\dots,D-n$, are hermitian scalar fields arising from the internal components of the gauge field and thus taking values in the adjoint representation of the gauge group, the covariant derivative is

$$D_\alpha\phi^a = \partial_\alpha\phi^a - i[A_\alpha,\phi^a] \quad , \tag{83}$$

A_α is the gauge field, $F_{\alpha\beta}$ its curvature, g_{YM}^2 denotes the Yang-Mills coupling constant, Tr a Lie algebra trace, and in writing the above action I have suppressed all Lie algebra labels.

This basic action can of course be modified in various ways, e.g. by adding further fields (we will not do this), or by modifying the couplings of the scalar fields. We will consider two such modifications. The first one simply consists of adding (possibly time-dependent) mass terms for the scalars. Denoting the scalars in this model by X^a, the action reads

$$\begin{aligned} S_{BC} = \int d^n\sigma\, \mathrm{Tr}\big(&-\tfrac{1}{4}g_{YM}^{-2}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} - \tfrac{1}{2}\eta^{\alpha\beta}\delta_{ab}D_\alpha X^a D_\beta X^b \\ &+\tfrac{1}{4}g_{YM}^2\delta_{ac}\delta_{bd}[X^a,X^b][X^c,X^d] + \tfrac{1}{2}A_{ab}(t)X^aX^b\big) \quad , \end{aligned} \tag{84}$$

with $A_{ab}(t)$ minus the mass-squared matrix.

The second class of actions arises from (82) by replacing the flat metric δ_{ab} on the scalar field space (suppressed in (82) but written out explicitly in (84)) by a time-dependent matrix $g_{ij}(t)$ of "coupling constants", but without adding any mass terms. Denoting the scalars in this model by Y^i, the action reads

$$\begin{aligned} S_{RC} = \int d^n\sigma\, \mathrm{Tr}\big(&-\tfrac{1}{4}g_{YM}^{-2}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} - \tfrac{1}{2}\eta^{\alpha\beta}g_{ij}(t)D_\alpha Y^i D_\beta Y^j \\ &+\tfrac{1}{4}g_{YM}^2 g_{ik}(t)g_{jl}(t)[Y^i,Y^j][Y^k,Y^l]\big) \quad . \end{aligned} \tag{85}$$

The reason for the subscripts $_{BC}$ and $_{RC}$ on the actions will, if not already obvious at this stage, become apparent below. In any case, the claim is now that these two, apparently rather different, classes of Yang-Mills actions are simply related by a certain linear field redefinition $Y^i = E^i_{\ a}X^a$ of the scalar fields,

$$S_{RC}[A_\alpha, Y^i = E^i_{\ a}X^a] = S_{BC}[A_\alpha, X^a] \quad . \tag{86}$$

We could straightaway prove this by a brute-force calculation, but this would be rather unenlightning. Instead, we will first consider a much simpler classical mechanics toy model of this equivalence (section 4.2), and we will then be able to establish (86) with hardly any calculation at all (section 4.3). At the end, I will briefly indicate why one is led to consider actions of the type (84,85) in the first place, and why from that point of view one can a priori anticipate the validity of an identity like (86).

4.2. A classical mechanics toy model

As a warm-up exercise, consider the standard harmonic oscillator Lagrangian (now mysteriously labelled $_{bc}$)

$$L_{bc}(x) = \tfrac{1}{2}(\dot{x}^2 - \omega^2 x^2) \quad , \tag{87}$$

and the (exotic) Lagrangian

$$L_{rc}(y) = \tfrac{1}{2}\sin^2\omega t\;\dot{y}^2 \tag{88}$$

with a time-dependent kinetic term. Now consider the transformation

$$y = (\sin\omega t)^{-1} x \quad . \tag{89}$$

Then one finds

$$\begin{aligned} L_{rc}(y) &= \tfrac{1}{2}(\dot{x}^2 + \omega^2 x^2 \cot^2\omega t - 2\omega x\dot{x}\cot\omega t) \\ &= \tfrac{1}{2}(\dot{x}^2 - \omega^2 x^2) - \tfrac{d}{dt}(\tfrac{1}{2}\omega x^2 \cot\omega t) \\ &= L_{bc}(x) + \tfrac{d}{dt}(\ldots) \end{aligned} \tag{90}$$

Thus, up to a total time-derivative the linear transformation (89) transforms the exotic (and seemingly somewhat singular) Lagrangian (88) to the completely regular "massive" Lagrangian (87), and the corresponding actions are essentially identical. This should be thought of as the counterpart of the statement that the Rosen coordinate plane wave metric

$$ds^2 = 2dUdV + \sin^2\omega U (dy)^2 \tag{91}$$

can, in Brinkmann coordinates, be written as

$$2dUdV + \sin^2\omega U(dy)^2 = 2dudv - \omega^2 x^2 (du)^2 + (dx)^2 \quad . \tag{92}$$

We can now generalise this in the following way. Consider the Lagrangian L_{bc} corresponding to the lightcone Hamiltonian (19) of a (massless, say) particle in a plane wave in Brinkmann coordinates (in the lightcone gauge $u = t$),

$$L_{bc}(x) = \tfrac{1}{2}(\delta_{ab}\dot{x}^a\dot{x}^b + A_{ab}(t)x^a x^b) \quad , \tag{93}$$

and the corresponding Lagrangian in Rosen coordinates,

$$L_{rc}(y) = \tfrac{1}{2} g_{ij}(t)\dot{y}^i\dot{y}^j \quad . \tag{94}$$

The claim is that these two Lagrangians are equal up to a total time-derivative. To see this, recall first of all the coordinate transformation (26)

$$\begin{aligned} y^i &= E^i{}_a x^a \\ V &= v + \tfrac{1}{2}\dot{E}_{ai}E^i{}_b x^a x^b \quad , \end{aligned} \tag{95}$$

where $E^i{}_a$ satisfies (24) and (25). Substituting $y^i = E^i{}_a x^a$ in L_{rc}, one can now verify that one indeed obtains L_{bc} up to a total time-derivative. The way to see this without any calculation is to start from the complete geodesic Lagrangian in Rosen coordinates in the lightcone gauge $U = t$,

$$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu}\dot{y}^\mu\dot{y}^\nu = \dot{V} + \tfrac{1}{2} g_{ij}(t)\dot{y}^i\dot{y}^j = \dot{V} + L_{rc}(y) \quad . \tag{96}$$

This Lagrangian is still invariant under coordinate transformations of the remaining coordinates, and is hence equal, on the nose, to its Brinkmannn coordinate counterpart (20),

$$L_{bc}(x) + \dot{v} = L_{rc}(y) + \dot{V} \quad . \tag{97}$$

This implies that the two Lagrangians $L_{bc}(x)$ and $L_{rc}(y)$ differ only by a total time-derivative, namely the derivative of the shift of V in the coordinate transformation (95).

4.3. The explanation: from plane wave metrics to Yang-Mills actions

We can now come back to the two types of Yang-Mills actions S_{BC} (84) and S_{RC} (85), which are obviously in some sense non-Abelian counterparts of the classical mechanics Brinkmann and Rosen coordinate actions actions $S_{bc} = \int L_{bc}$ and $S_{rc} = \int L_{rc}$ discuussed above. The claim is that these two actions are related (perhaps up to a total derivative term) by the linear transformation

$$Y^i = E^i_{\ a} X^a \tag{98}$$

of the scalar fields (matrix-valued coordinates) Y^i and X^a, where $E^i_{\ a}$ is the vielbein that enters in the relation between Rosen and Brinkmann coordinates.

Even though in general non-Abelian coordinate transformations are a tricky issue, this particular transformation is easy to deal with since it is linear as well as diagonal in matrix (Lie algebra) space. Consider e.g. the quartic potential terms in (84) and (85). With the substitution (98), one obviously has

$$\begin{aligned} g_{ik}g_{jl}[Y^i, Y^j][Y^k, Y^l] &= g_{ik}g_{jl}E^i_{\ a}E^j_{\ b}E^k_{\ c}E^l_{\ d}[X^a, X^b][X^c, X^d] \\ &= \delta_{ac}\delta_{bd}[X^a, X^b][X^c, X^d] \quad , \end{aligned} \tag{99}$$

so that the two quartic terms are indeed dirctly related by (98). Now consider the gauge covariant kinetic term for the scalars in (85). Since $E^i_{\ a} = E^i_{\ a}(t)$ depends only on (lightcone) time t, the spatial covariant derivatives transform as

$$\alpha \neq t : \quad D_\alpha Y^i = E^i_{\ a}(t) D_\alpha X^a \quad , \tag{100}$$

so that the spatial derivative parts of the scalar kinetic terms are mapped into each other. It thus remains to discuss the term $\mathrm{Tr}\, g_{ij}(t) D_t Y^i D_t Y^j$ involving the covariant time-derivatives. For the ordinary partial derivatives, the argument is identical to that in section 4.2, and thus one finds

$$\tfrac{1}{2}\,\mathrm{Tr}\, g_{ij}(t)\dot{Y}^i\dot{Y}^j = \tfrac{1}{2}\,\mathrm{Tr}(\delta_{ab}\dot{X}^a\dot{X}^b + A_{ab}(t)X^aX^b) + \tfrac{d}{dt}(\ldots) \quad . \tag{101}$$

The only remaining subtlety are terms involving the t-derivative $\dot{E}^i_{\ a}$ of $E^i_{\ a}$, arising from cross-terms like

$$\mathrm{Tr}\, g_{ij}(t)[A_t, Y^i]\partial_t Y^j = \mathrm{Tr}\, g_{ij}(t)E^i_{\ a}[A_t, X^a]\partial_t(E^j_{\ b}X^b) \quad . \tag{102}$$

However, these terms do not contribute at all since

$$g_{ij}(t)E^i_{\ a}\dot{E}^j_{\ b}\,\mathrm{Tr}[A_t, X^a]X^b = g_{ij}(t)E^i_{\ a}\dot{E}^j_{\ b}\,\mathrm{Tr}\, A_t[X^a, X^b] = 0 \tag{103}$$

by the cyclic symmetry of the trace and the symmetry condition (25). It is pleasing to see that this symmetry condition, which already ensured several cancellations in the standard tranformation from Rosen to Brinkmann cooordinates (and thus also in establishing e.g. (101)), cooperatively also serves to eliminate some terms of genuinely non-Abelian origin.

Putting everything together, we have now established the claimed equivalence (86) between the two apparently quite different classes of Yang-Mills theories, namely standard Yang-Mills theories with (possibly time-dependent) mass-terms on the one hand, and Yang-Mills theories with non-standard time-dependent scalar couplings on the other.

I still owe you an explanation of where all of this comes from or what it is good for. The appropriate context for this is provided by a non-perturbative description of type IIA string theory in certain backgroounds known as *matrix string theory* [15]. In this context, the standard action (82), with $p = 1$ and $D = 10$, suitably supersymmetrised, and with gauge group $U(N)$, describes IIA string theory in a Minkowski background. The Yang-Mills coupling constant g_{YM} is inversely related to the string coupling constant g_s. At weak string (strong gauge) coupling, the quartic term forces the non-Abelian coordinates ϕ^a, with $a = 1, \ldots, 8$, to commute, so that they can be considered as ordinary coordinates. One can show that (oversimplifying things a bit, since this should really be thought of as a second quantised description) in this limit one reproduces the usual weak coupling lightcone quantisation of the string.

However, the description of string theory based on the action (82) is equally well defined at strong string (weak gauge) coupling, where the full non-Abelian dynamics of the gauge theory becomes important. This is one indication that at strong coupling the target space geometry of a string may be described by a very specific kind of (matrix) non-commutative geometry.

The generalised actions (84,85) arise in the matrix string description of strings propagating in plane wave backgrounds, and the general covariance of this description leads one to a priori expect a relation of the kind (86). These kinds of models, generalisations of the Matrix Big Bang model of [16], become particularly interesting for singular plane waves with a singularity at strong string coupling (so that a perturbative string description is obviously inadequate), and one can investigate what the non-Abelian dynamics (non-commutative geometry) says about what happens at such a space-time singularity. Some of these issues will be explored in [3], from which also the entire discussion of this section 4 is taken.

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