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Killing vectors, constraints and conserved charges in Kaluza-Klein theories

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Abstract. The traditional Abelian and non-Abelian Kaluza-Klein ansätze are analysed from a geometrical point of view using a Bochner-Yano type theorem for geodesic Killing vector fields. It is shown how the presence of Killing vectors in the ansatz gives rise to constraints on the Yang-Mills fields and the curvature. The conserved Komar quantities associated with these Killing vectors are computed and compared with the corresponding ADM charges. A crucial difference between them in the presence of external fields is pointed out.

1. Introduction

In the quest for a theory unifying gravity with the other fundamental interactions, two main lines of attack have been pursued. One may try to interpret gravity as a gauge theory in order to make contact with the description of electroweak and strong interactions as $U(1)\times SU(2)$ and $SU(3)$ gauge theories. But quite apart from the fact that there is not even a general agreement on what the gauge group of gravity actually should be (translations [1], de Sitter [2], Poincaré [3] or Lorentz [4]), there are two main obstacles facing this approach (which are in fact related): on the one hand, conventional Yang-Mills (YM) type Lagrangians are quadratic in the field strengths and would therefore naturally lead to R^2 type Lagrangians for gravity. On the other hand, the existence of a soldering form [5, 6]—which allows 'internal' and 'spacetime' indices of the curvature tensor to be contracted and therefore permits the construction of a linear Lagrangian (as opposed to YM theories)—reduces the automorphism group of the frame bundle of the manifold M to purely 'horizontal' diffeomorphisms of M [6 p 226, 7]. This of course renders a gauging of 'vertical' transformations (as in conventional gauge theories) meaningless.

Since it therefore seems difficult to interpret gravity as a gauge theory one may turn the idea around and try to incorporate gauge interactions into gravity by realising gauge symmetries as spacetime symmetries in higher dimensions. This has (*cum grano salis*, see below) been shown to be possible for electromagnetism by Kaluza [8] and Klein [9] some 60 years ago. Consequent developments which mainly aimed at a non-Abelian generalisation of the original model [10] received an upsurge in interest

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in the context of Kaluza-Klein supergravity (a complete account may be found in [11]) and Kaluza-Klein cosmology [12]. Even more recently Duff *et al* [13] have shown that the heterotic string has a Kaluza-Klein origin in 506 ($=10+\dim E8\times E8$) dimensions.

If one regards the extra D dimensions as real—which is the generally accepted philosophy by now—one should also take the full set of $(D+4)$-dimensional Einstein equations seriously. The fact that this may lead to consistency problems in the standard Kaluza-Klein models has first been addressed by Duff *et al* [14], who pointed out that in general solutions of the truncated four-dimensional effective theory may not be solutions of the $(D+4)$-dimensional field equations (as required).

Assuming—as we shall do in our models of § 2—that the manifold M is of the form $M_4\times G$ (four-dimensional spacetime $\times$ non-Abelian group manifold), consistency of the spacetime components of the field equations may be restored by keeping only the $G_{\rm L}$ (or $G_{\rm R}$) bosons of the full $G_{\rm L}\times G_{\rm R}$ symmetry of the internal space. Our ansatz (3.22) will take this into account. However, even in the Abelian case another difficulty arises: the 'internal' (3.21) component of the field equation imposes a severe constraint ($F_{\mu\nu}F^{\mu\nu}=0$) on the field strength which may only be weakened by the inclusion of a scalar field corresponding to the 55-component of the metric. Both alternatives show that Einstein-Maxwell is (contrary to common belief) not a consistent truncation of five-dimensional gravity (this is the grain of salt alluded to above). Similar remarks apply to Einstein-YM theory, where the field equations again impose severe constraints on the field strengths ($F_a^{\mu\nu}F_{b\mu\nu}\sim g_{ab}$).

In this paper we shall show that these constraints are a direct consequence of the existence of Killing vectors of the full $(4+D)$-dimensional metric (which are not the Killing vectors of the internal metric generating gauge transformations). In order to do this we shall derive a Bochner-Yano [15] type theorem for pseudoRiemannian manifolds and apply this to our Kaluza-Klein spacetimes.

A further consequence of the existence of Killing vectors is the possibility to define conserved currents (a generalisation of Komar's [16] conserved quantities) and charges. Upon comparison of these charges with those obtained via a suitable modification of the usual ADM definition [17] for energy and momentum it turns out that the so-defined ADM charges are conserved, even when the YM field is coupled with other fields, whereas the Komar charge is not, since it fails to take into account the possible exchange of charge between the YM field and the other fields coupled to it. This observation seems to invalidate the suggestion, sometimes made in the literature, that the Komar quantities associated with timelike and spacelike Killing vectors should be identified with energy and momentum even in non-source-free regions of spacetime.

More precisely this paper is organised as follows. In § 1 we recall basic concepts of the geometry of Lie groups which will be used throughout this work. Section 2 consists of two parts, in which the Abelian and non-Abelian Kaluza-Klein ansätze are discussed respectively. We compute the connection and curvature forms, obtain the field equations and focus particular attention on the appearance of the constraints mentioned above.

In order to understand their geometrical significance, we shall derive a Bochner-Yono type theorem in § 3 which allows us to arrive at these constraints on grounds of purely geometrical considerations in § 4. In § 5 we show how Killing vectors give rise to closed $(n-2)=(D+2)$-forms [18] and compute their associated charges. Based on the observation that the Landau-Lifshitz pseudotensor is not only a closed but an exact form, we arrive directly at the ADM energy-momentum in § 6 without performing a canonical $(3+1)$ or $(3+D+1)$ decomposition; then we compare the charges thus obtained (the 'internal' components of the ADM momentum) with the Komar charges of § 5 and find a crucial difference in the presence of external fields.

The material of §§ 2 and 3 is based on joint work with Landi and Thirring and has mainly been taken from [19], where details of some of the calculations may be found.

2. Geometry of Lie groups

To begin with we shall—to fix the notation and mathematical setting—recall some basic facts on the geometry of Lie groups. Let G be a compact semisimple Lie group and denote by L(G) its Lie algebra with basis t_a, $a=1,\ldots,D=\dim \mathrm{G}$. The t_a satisfy

$$[t_a, t_b]=C^c{}_{ab}t_c \tag{2.1}$$

where the $C^c{}_{ab}$ are the structure constants of G obeying the Jacobi identity

$$C^a{}_{bd}C^d{}_{ef}+C^a{}_{fd}C^d{}_{be}+C^a{}_{ed}C^d{}_{fb}=0. \tag{2.2}$$

The left- (right-) invariant Maurer–Cartan forms θ^a and $\hat{\theta}^a$ on G are defined by

$$s^{-1}\,\mathrm{d}s=\theta^a(s)t_a \tag{2.3}$$

and

$$\mathrm{d}s\,s^{-1}=-\hat{\theta}^a(s)t_a. \tag{2.4}$$

Their dual left- (right-) invariant vector fields Y_a and $\hat{Y}_a$

$$i_{Y_a}\theta^b=\delta_a{}^b=i_{\hat{Y}_a}\hat{\theta}^b \tag{2.5}$$

(where i is the interior product or contraction) are the generators of right (left) translations respectively. Denoting by d the exterior and $\mathrm{L}(=i_0\mathrm{d}+\mathrm{d}_0 i$ on forms) the Lie derivative, we immediately arrive at the following relations:

$$i_{Y_a}(s^{-1}\,\mathrm{d}s)=t_a \tag{2.6}$$

$$i_{\hat{Y}_a}(\mathrm{d}s\,s^{-1})=-t_a \tag{2.7}$$

$$\mathrm{d}\theta^a=-\tfrac{1}{2}C^a{}_{bc}\theta^b\wedge\theta^c \tag{2.8}$$

$$\mathrm{d}\hat{\theta}^a=-\tfrac{1}{2}C^a{}_{bc}\hat{\theta}^b\wedge\hat{\theta}^c \tag{2.9}$$

$$\mathrm{L}_{Y_b}\theta^a=-C^a{}_{bc}\theta^c \tag{2.10}$$

$$\mathrm{L}_{\hat{Y}_b}\hat{\theta}^a=-C^a{}_{bc}\hat{\theta}^c \tag{2.11}$$

$$\mathrm{L}_{Y_a}\hat{Y}_b=[Y_a,\hat{Y}_b]=0 \tag{2.12}$$

$$\mathrm{L}_{\hat{Y}_a}Y_b=[\hat{Y}_a,Y_b]=0 \tag{2.13}$$

$$\mathrm{L}_{Y_a}Y_b=[Y_a,Y_b]=C^c{}_{ab}Y_c \tag{2.14}$$

$$\mathrm{L}_{\hat{Y}_a}\hat{Y}_b=[\hat{Y}_a,\hat{Y}_b]=C^c{}_{ab}\hat{Y}_c. \tag{2.15}$$

The matrices $D^b{}_a(S)$ of the adjoint representation of G on L(G) are defined by

$$Ad(s)t_a=: D^b{}_a(s)t_b. \tag{2.16}$$

They allow us to relate the left- and right-invariant forms and vector fields on G with each other:

$$\hat{\theta}^a = -\theta^b D^a{}_b(s) \tag{2.17}$$

$$\theta^a = -\hat{\theta}^b D^a{}_b(s^{-1}) \tag{2.18}$$

$$\hat{Y}_a = -Y_b D^b{}_a(s^{-1}) \tag{2.19}$$

$$Y_a = -\hat{Y}_b D^b{}_a(s). \tag{2.20}$$

From (2.13) and (2.20) we can compute $\mathrm{d}D^a{}_b(s)$

$$0 = \mathrm{L}_{\hat{Y}_a} Y_b = -C^d{}_{ac} D^c{}_b(s)\hat{Y}_d - i_{\hat{Y}_a}\, \mathrm{d}D^c{}_b(s)\hat{Y}_d$$

$$\Rightarrow \mathrm{d}D^a{}_b(s) = -C^c{}_{bd} D^a{}_c(s)\theta^d \tag{2.21}$$

and analogously in terms of $\hat{\theta}^a$

$$\mathrm{d}D^a{}_b(s) = -C^a{}_{dc} D^c{}_b(s)\hat{\theta}^d. \tag{2.22}$$

These relations allow us to complete the set of equations (2.10)-(2.15) by

$$\mathrm{L}_{Y_a}\hat{\theta}^b = 0 \tag{2.23}$$

$$\mathrm{L}_{\hat{Y}_a}\theta^b = 0. \tag{2.24}$$

Since we assumed G to be semisimple and compact, the bi-invariant Killing-Cartan form

$$g_{ab} = -C^c{}_{ad} C^d{}_{bc} \tag{2.25}$$

is non-degenerate and positive definite. Given any metric $g = g_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$ on G, the left-invariant vector fields Y_a are of constant length iff g is Ad-invariant, since

$$\begin{aligned} g(Y_a, Y_b) &= g_{cd} D^c{}_a(s) D^d{}_b(s) \\ &= (Ad(s)g)_{ab}. \end{aligned} \tag{2.26}$$

This remark will turn out to be useful in the context of Kaluza-Klein theories, which we shall discuss in the next section.

For the connection forms $\omega^a{}_b$ on the group manifold G, restricted by metricity

$$\mathrm{d}g_{ab} = \omega_{ab} + \omega_{ba} = 0 \tag{2.27}$$

and defining the torsion

$$\tau^a = \mathrm{d}\theta^a + \omega^a{}_b \wedge \theta^b \tag{2.28}$$

three interesting choices are obtained by regarding

$$\omega^a{}_b = KC^a{}_{cb}\theta^c \qquad K \in \mathbb{R}. \tag{2.29}$$

Computing the torsion and curvature of (2.29) we obtain

$$\tau^a = (K - \tfrac{1}{2})C^a{}_{bc}\theta^b \wedge \theta^c \tag{2.30}$$

and

$$\Omega^a{}_b = -\tfrac{1}{2}K(K-1)C^a{}_{bc}C^c{}_{de}\theta^d \wedge \theta^e. \tag{2.31}$$

This shows that the choices $K=0$ and $K=1$ (the $(-)$ and $(+)$ connections of [6]) amount to introducing a parallelising torsion (2.30) on G, whereas the case $K=\frac{1}{2}$ corresponds to the uniquely determined metric (2.27) and torsion-free (2.30) Levi-Civita connection on G, which is, however, not flat (2.31). For $K=\frac{1}{2}$ the Ricci forms are given by

$$\Omega_b = i_a \Omega^a{}_b = \tfrac{1}{4} g_{be} \theta^e \tag{2.32}$$

and the Ricci scalar is

$$\Omega = i^b \Omega_b = \tfrac{1}{4} D. \tag{2.33}$$

Therefore torsion-free group manifolds are Einstein spaces (2.32) with strictly positive scalar curvature (2.33) or every group manifold admits a (left- and right-invariant) Einstein metric. Due to our choice of sign in (2.4) all the equations remain unaltered if the left-invariant Maurer-Cartan forms θ^a are replaced by the right-invariant $\hat{\theta}^a$.

3. Kaluza–Klein ansätze and constraints

As announced in § 1, we shall in this section restrict ourselves to a discussion of some mathematical consequences of taking the extra dimensions (and hence the full set of higher-dimensional vacuum Einstein equations) seriously. This implies that we adhere to the Einstein–Kaluza point of view, that all matter should arise from geometry and no fields should be added by hand. Thus, for the time being, the question of how fermionic matter should enter the game remains unanswered. We shall, however, leave these considerations aside here and concentrate on the bosonic sector of Kaluza-Klein theories.

Let us start by looking at the original five-dimensional theory [8, 9]. The ansatz for the metric on $M_5 = M_4 \times S^1$ ($M_n = n$-dimensional manifold) is

$$g = g_{\mu\nu} e^\mu \otimes e^\nu + e^5 \otimes e^5 \tag{3.1}$$

with

$$e^\mu = \mathrm{d}x^\mu \qquad \mu = 0, 1, 2, 3 \tag{3.2}$$

and

$$e^5 = \mathrm{d}x^5 + A = \mathrm{d}x^5 + A_\mu(x^\nu)\, \mathrm{d}x^\mu. \tag{3.3}$$

The dual basis is given by

$$\tilde{e}_\mu = \partial_\mu - A_\mu \partial_5 \tag{3.4}$$

and

$$\tilde{e}_5 = \partial_5 \qquad (\partial_A = \partial/\partial x^A). \tag{3.5}$$

$\tilde{e}_5$ is obviously a Killing vector field of g and since it is moreover of constant length it is also geodesic (cf 4.3). Since the change of the metric under coordinate transformations generated by vector fields of the form $f(x^\mu)\tilde{e}_5$ corresponds to gauge transformations of A, $A \to A - \mathrm{d}f$, the general covariance of the five-dimensional Einstein theory implies the local U(1) symmetry of electromagnetism. To see that the Einstein Lagrangian in five dimensions really yields the Einstein-Maxwell Lagrangian in the more familiar four dimensions (the so-called 'Kaluza-Klein miracle'), let us compute the Levi-Civita connection and curvature forms determined by (2.27) and (2.28) with $\tau^A = 0$ ($A = 0, 1, 2, 3, 5$; $\mu = 0, 1, 2, 3$):

$$\omega^\rho{}_\mu = {}^{(4)}\omega^\rho{}_\mu - \tfrac{1}{2} F^\rho{}_\mu e^5 \tag{3.6}$$

$$\omega^5{}_\mu = \tfrac{1}{2} i_\mu F \tag{3.7}$$

$$\omega^\rho{}_5 = -\tfrac{1}{2} i^\rho F \tag{3.8}$$

$$\omega^5{}_5 = 0 \tag{3.9}$$

where ${}^{(4)}\omega^\rho{}_\mu$ are the four-dimensional Levi-Civita connection forms of $g_{\mu\nu}$ and $F = \mathrm{d}A = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

From Cartan's second structure equation

$$\Omega^A{}_B = \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{3.10}$$

for the curvature forms we obtain

$$\begin{aligned}\Omega^\rho{}_\mu = {}& \mathrm{d}\,{}^{(4)}\omega^\rho{}_\mu + \tfrac{1}{2}\mathrm{d}(F_\mu{}^\rho) \wedge e^5 + \tfrac{1}{2} F_\mu{}^\rho F + {}^{(4)}\omega^\rho{}_\nu \wedge {}^{(4)}\omega^\nu{}_\mu \\ & + \tfrac{1}{2}{}^{(4)}\omega^\rho{}_\nu \wedge F_\mu{}^\nu e^5 + \tfrac{1}{2} F_\nu{}^\rho e^5 \wedge {}^{(4)}\omega^\nu{}_\mu - \tfrac{1}{4} i^\rho F \wedge i_\mu F\end{aligned} \tag{3.11}$$

$$\begin{aligned}\Omega^5{}_\mu &= -\Omega_\mu{}^5 \\ &= \tfrac{1}{2}\mathrm{d} i_\mu F + \tfrac{1}{2} i_\nu F \wedge {}^{(4)}\omega^\nu{}_\mu + \tfrac{1}{4} i_\nu F \wedge F_\mu{}^\nu e^5.\end{aligned} \tag{3.12}$$

Using [19]

$$\overset{4}{*}\,\mathrm{d}\,\overset{4}{*}\, F = -i^\mu(\mathrm{d} i_\mu F + i_\nu F \wedge {}^{(4)}\omega^\nu{}_\mu) \tag{3.13}$$

and

$$i_\mu \overset{4}{*}\,\mathrm{d}\,\overset{4}{*}\, F = i_\rho(\mathrm{d}F_\mu{}^\rho + {}^{(4)}\omega^\rho{}_\nu F_\mu{}^\nu + {}^{(4)}\omega_\mu{}^\nu F_\nu{}^\rho) \tag{3.14}$$

where $\overset{4}{*}$ is the four-dimensional Hodge duality operator, we find for the Ricci forms

$$\Omega_B = i_A \Omega^A{}_B$$

$$\Omega_5 = \tfrac{1}{4} F_{\mu\rho} F^{\mu\rho} e^5 + \tfrac{1}{2} \overset{4}{*}\,\mathrm{d}\,\overset{4}{*}\, F \tag{3.15}$$

$$\Omega_\mu = {}^{(4)}\Omega_\mu + \tfrac{1}{2} F_{\mu\rho} i^\rho F + \tfrac{1}{2}(i_\mu \overset{4}{*}\,\mathrm{d}\,\overset{4}{*}\, F) e^5 \tag{3.16}$$

and finally for the Ricci scalar $\Omega = i^B \Omega_B$

$$\Omega = {}^{(4)}\Omega - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.17}$$

Equation (17) is (as announced) the conventional Einstein-Maxwell Lagrangian, and varying $\int \overset{5}{*}\, \Omega$ with respect to $g_{\mu\nu}$ and A indeed leads to the coupled Einstein-Maxwell equations

$${}^{(4)}\Omega^\nu - \tfrac{1}{2} e^{\nu}\, {}^{(4)}\Omega = T^\nu{}_\mu e^\mu \tag{3.18}$$

$$\mathrm{d}\,\overset{4}{*}\, F = 0 \tag{3.19}$$

where $T^\nu{}_\mu$ is the energy-momentum tensor of the electromagnetic field. The embarrassing observation is that the set of equations (3.18) and (3.19) is not equivalent to the five-dimensional Einstein equations

$$\Omega^A - \tfrac{1}{2} e^A \Omega = 0. \tag{3.20}$$

In addition to (3.18) and (3.19) they demand

$$F_{\mu\rho} F^{\mu\rho} = -2\,\overset{4}{*}\,(F \wedge \overset{4}{*}\, F) = 0. \tag{3.21}$$

Moreover, using (3.17) or (3.18), (3.21) constrains the four-dimensional Ricci scalar to be zero, ${}^{(4)}\Omega=0$, which (although desirable) does not persist in the non-Abelian case where we instead have ${}^{(4)}\Omega<0$ (cf 3.36). This additional constraint on the Maxwell field arises, because varying only with respect to $g_{\mu\nu}$ and A amounts to keeping g_{55} fixed, while the (more restrictive) Einstein equations are of course obtained upon variation with respect to the full five-dimensional metric. We shall see later (§ 4) that the constraint (3.21) is intimately related to the existence of a Killing vector in the ansatz (3.1). Before exhibiting the precise relationship, we shall show that the same problem arises in non-Abelian Kaluza-Klein theories. Using the notation of § 2, we write the $n=(D+4)$-dimensional metric as [19, cf also 20]

$$g=g_{\mu\nu}\,\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}+g_{ab}T^{a}\otimes T^{b} \tag{3.22}$$

with

$$e^{A}=(\mathrm{d}x^{\mu},\,T^{a})=(\mathrm{d}x^{\mu},\,\hat{\theta}^{a}-A^{a}(x^{\nu})) \tag{3.23}$$

and its dual

$$\tilde{e}_{A}=(\partial_{\mu}+A^{b}_{\mu}\hat{Y}_{b},\,\hat{Y}_{a}) \tag{3.24}$$

and

$$g_{ab}=-C^{c}{}_{ad}C^{d}{}_{bc}.$$

Spelled out in terms of the Killing vectors $\hat{Y}_a$ instead of the Maurer-Cartan form $\hat{\theta}^a$, (3.22) is seen to coincide with the traditional non-Abelian ansatz. As a consequence of (2.23) the left-invariant fields Y_a are Killing vectors of (3.22) and (2.26) tells us that they are of constant length and therefore additionally geodesic (4.3). The special ansatz (3.22) has been chosen, because it displays the desirable feature that gauge transformations of A^a correspond to a special kind of coordinate transformations (cf the remarks following (3.5)). Choose $X=\varepsilon^{a}(x^{\mu})\hat{Y}_a$ for some arbitrary functions $\varepsilon^{a}(x^{\mu})$; it then follows that

$$\begin{aligned}(\mathrm{L}_{X}g)_{\mu b}&=\partial_{\mu}\varepsilon_{b}+C_{bac}A^{a}{}_{\mu}\varepsilon^{c}\\&=\mathrm{D}_{\mu}\varepsilon_{b}=-\delta A_{b\mu}\end{aligned} \tag{3.25}$$

Here D is the G-covariant derivative and δ an infinitesimal gauge transformation. Since the right-invariant vector fields $\hat{Y}_a$ generate left translations, (3.25) may also be equivalently restated in the following way. g is invariant under simultaneous transformation of $\hat{\theta}^a$ under left translation and a gauge transformation of A^a (had we not chosen g_{ab} to be Ad-invariant, we would have been forced to transform g_{ab} as well). In a straightforward generalisation of equations (3.6)–(3.11) the connection and curvature forms of g are found to be [19]

$$\omega^{\mu}{}_{\nu}={}^{(4)}\omega^{\mu}{}_{\nu}+\tfrac{1}{2}F_{a}{}^{\mu}{}_{\nu}T^{a} \tag{3.26}$$

$$\omega^{a}{}_{\mu}=\tfrac{1}{2}i_{\mu}F^{a} \tag{3.27}$$

$$\omega^{a}{}_{b}=\tfrac{1}{2}C^{a}{}_{cb}T^{c}+C^{a}{}_{cb}A^{c} \tag{3.28}$$

(with the Yang-Mills field strength $F^{a}=\mathrm{d}A^{a}+\tfrac{1}{2}C^{a}{}_{bc}A^{b}\wedge A^{c}$).

$$\begin{aligned}\Omega^{\mu}{}_{\nu}={}&{}^{(4)}\Omega^{\mu}{}_{\nu}-\tfrac{1}{2}\,\mathrm{d}(F_{a\nu}{}^{\mu})\wedge T^{a}+\tfrac{1}{2}F^{a}{}_{\nu}{}^{\mu}F_{a}+\tfrac{1}{4}F^{a}{}_{\nu}{}^{\mu}C_{abc}T^{b}\wedge T^{c}\\&+\tfrac{1}{2}F^{a}{}_{\nu}{}^{\mu}C_{abc}A^{b}\wedge T^{c}+\tfrac{1}{2}F_{a\rho}{}^{\mu}\,{}^{(4)}\omega^{\rho}{}_{\nu}\wedge T^{a}-\tfrac{1}{2}\,{}^{(4)}\omega^{\mu}{}_{\rho}\wedge T^{a}F_{a\nu}{}^{\rho}\\&+\tfrac{1}{4}F_{a\rho}{}^{\mu}F_{b\nu}{}^{\rho}T^{a}\wedge T^{b}-\tfrac{1}{4}(i^{\mu}F_{a})\wedge i_{\nu}F^{a}\end{aligned} \tag{3.29}$$

$$\Omega^a{}_\nu = -\tfrac{1}{2}\,\mathrm{d}i_\nu F^a - \tfrac{1}{4}C^a{}_{cb}T^c \wedge i_\nu F^b - \tfrac{1}{2}C^a{}_{cb}A^c \wedge i_\nu F^b$$
$$-\tfrac{1}{2}(i_\mu F^a) \wedge {}^{(4)}\omega^\mu{}_\nu + \tfrac{1}{4}(i_\mu F^a) \wedge F_{b\nu}{}^\mu T^b \tag{3.30}$$

$$\Omega^a{}_b = -\tfrac{1}{8}C^a{}_{cb}C^c{}_{de}T^d \wedge T^e + \tfrac{1}{2}C^a{}_{cb}F^c - \tfrac{1}{4}(i_\mu F^a) \wedge (i^\mu F_b). \tag{3.31}$$

Writing the non-linear part of the G-covariant derivative $\mathrm{D} \overset{4}{*} F_a$ as

$$(A \wedge \overset{4}{*} F)_a := C_{abc}A^b \wedge \overset{4}{*} F^c$$

the Ricci forms are

$$\Omega_\nu = {}^{(4)}\Omega_\nu + \tfrac{1}{2}F^a{}_\nu{}^\mu i_\mu F_a - \tfrac{1}{2}i_\nu \overset{4}{*}\, \mathrm{D} \overset{4}{*} F_a \wedge T^a \tag{3.32}$$

$$\Omega_b = \tfrac{1}{4}(F_{a\mu\nu}F_b{}^{\mu\nu} + g_{ab})T^a - \tfrac{1}{2} \overset{4}{*}\, \mathrm{D} \overset{4}{*} F_b \tag{3.33}$$

and the Ricci scalar yields the Einstein-Yang-Mills (EYM) Lagrangian with a cosmological constant $\frac{1}{4}\mathrm{D}$ which arises, since the internal space is not flat (like the $S^1 \simeq \mathrm{U}(1)$ was in the Abelian case)

$$\Omega = {}^{(4)}\Omega - \tfrac{1}{4}F_{a\mu\nu}F^{a\mu\nu} + \tfrac{1}{4}D. \tag{3.34}$$

Equation (3.34) is at the heart of the so-called cosmological constant problem [21] since $\frac{1}{4}D$ is roughly 120 orders of magnitude too large to be physically acceptable. Thus—in order to make this term disappear from four-dimensional physics—one ought to solve the Einstein equations with a (fine-tuned) cosmological constant.

Insisting, however, on pure Einstein gravity in $(D+4)$ dimensions and varying $\int \overset{n}{*}\, \Omega$ with respect to $g_{\mu\nu}$ and A^b leads to the coupled EYM equations (now unfortunately with a cosmological constant), whereas the $(D+4)$-dimensional Einstein equations additionally demand

$$F_{a\mu\nu}F_b{}^{\mu\nu} = -g_{ab} \tag{3.35}$$

restricting the scalar curvature ${}^{(4)}\Omega$ to

$${}^{(4)}\Omega = -\tfrac{1}{2}D < 0. \tag{3.36}$$

Of course the constraints (3.21) and (3.35) should be eliminated by introducing $\frac{1}{2}D(D+1)$ (so far unobserved) Brans-Dicke-type scalar fields corresponding to the unvaried components of the metric g_{55} or g_{ab}; but despite some of their virtues as possible Higgs or inflaton fields [22] they usually turn out to be more of a nuisance, because they lead to a non-polynomial Lagrangian and couple non-minimally to gravity and/or YM gauge fields (the question of which fields may consistently be dropped or which have to be kept to ensure compatibility with the full set of Einstein's equations has been analysed and clarified by Duff and his collaborators in their 'Kaluza-Klein consistency' programme [11, 14]).

However another unpleasant feature of (3.35) is that this condition breaks the Lorentz invariance of the ground state† which already disqualifies the conventional Kaluza-Klein ansatz at the classical level.

4. Killing vectors and a Bocher-Yano type theorem

Since symmetries (and therefore the existence of Killing vectors) are crucial for the emergence of local gauge invariance after compactification, we shall address the

† This had kindly been pointed out to me by the referee.

implication of the existence of Killing vectors in a very general setting (arbitrary $(D+4)$-dimensional pseudoRiemannian manifold M), and only later restrict ourselves to compact or Riemannian Einstein spaces and to geodesic Killing vector fields to make contact with the models of § 3.

Given a metric g on M, a vector field X on M is Killing if it satisfies

$$L_X g = 0. \tag{4.1}$$

Denoting by $\langle\ ,\ \rangle$ the scalar product determined by g and by ∇ its Levi-Civita connection, (4.1) is equivalent to

$$\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0 \qquad \forall\, Y, Z. \tag{4.2}$$

Therefore, especially, covariantly constant vector fields are Killing. Furthermore, a Killing vector field X is geodesic (i.e. $\nabla_X X = 0$) iff it is of constant length, since

$$\tfrac{1}{2}\nabla_Y\langle X, X\rangle = \langle \nabla_Y X, X\rangle = -\langle Y, \nabla_X X\rangle \qquad \forall\, Y. \tag{4.3}$$

Using the fact that L_X and ∇ commute if X is Killing, and introducing a basis of 1-forms e^A (e.g. (3.23)) and its dual $\tilde{e}_A$ (e.g. (3.24)), it may be shown [19] that X satisfies

$$(\nabla^A\nabla_A - \nabla_{\nabla^A \tilde{e}_A})X = -(i_X\Omega^A)\tilde{e}_A \tag{4.4}$$

(with $\nabla^A = g^{AB}\nabla_B$ and $\nabla_A = \nabla_{\tilde{e}_A}$). Taking the scalar product with X on both sides of (4.4) leads to

$$\tfrac{1}{2}(\nabla^A\nabla_A - \nabla_{\nabla^A\tilde{e}_A})\langle X, X\rangle - \langle \nabla_A X, \nabla^A X\rangle = -i_X\Omega_A X^A. \tag{4.5}$$

Since the first term on the left-hand side may be written as $-* \mathrm{d} * \mathrm{d}\langle X, X\rangle$ we arrive at the very useful Bochner-Yano [15] formula

$$\tfrac{1}{2} * \mathrm{d} * \mathrm{d}\langle X, X\rangle + \langle \nabla_A X, \nabla^A X\rangle = i_X\Omega_A X^A \tag{4.6}$$

valid for any Killing vector field X on a (pseudo)Riemannian manifold M.

If we now assume that M is a solution of the $(D+4)$-dimensional Einstein field equations with cosmological constant Λ, i.e.

$$\Omega^A - \tfrac{1}{2}e^A\Omega + e^A\Lambda = 0 \tag{4.7}$$

or equivalently

$$\Omega^A = 2\Lambda(D+2)^{-1}e^A \tag{4.8}$$

then (4.6) reduces to

$$\tfrac{1}{2} * \mathrm{d} * \mathrm{d}\langle X, X\rangle + \langle \nabla_A X, \nabla^A X\rangle = 2\Lambda(D+2)^{-1}\langle X, X\rangle. \tag{4.9}$$

Under the additional assumption that M is compact and without boundary, we can integrate (4.9) over M, drop the first term, and obtain

$$\int_M (\langle \nabla_A X, \nabla^A X\rangle - 2\Lambda(D+2)^{-1}\langle X, X\rangle) * 1 = 0. \tag{4.10}$$

This is precisely the Yano formula (equation 2.71 of [15]) which in the Riemannian case implies that there are no Killing vector fields for $\Lambda < 0$ and that for $\Lambda = 0$ every Killing vector has to be covariantly constant.

Applying (4.10) to the internal manifold of Kaluza-Klein models (which are usually assumed to be Riemannian) leads to the conclusion that Ricci flatness implies the existence of at most Abelian continuous symmetries (since $[X, Y]=\nabla_X Y-\nabla_Y X=0$ for two covariantly constant Killing vector fields); this is of course disastrous if one wants to obtain interactions other than electromagnetism via the Kaluza-Klein mechanism.

In the situation in which we are interested, we can arrive at similar conclusions without invoking any integration, without restricting M to be compact, and without specifying any asymptotic behaviour of X. We make use of the fact that the Killing vector fields we have encountered so far are also geodesic. In those cases we can drop the first term on the left-hand side of (4.9) right away to obtain

$$\langle\nabla_A X, \nabla^A X\rangle=2\Lambda(D+2)^{-1}\langle X, X\rangle. \tag{4.11}$$

We see that if M is Riemannian, there are no geodesic Killing vector fields if $\Lambda<0$ and at most geodesic Abelian symmetries if $\Lambda=0$.

In the case of pseudoRiemannian Ricci-flat manifolds (e.g. our Kaluza-Klein n manifolds, if we demand them to satisfy the Einstein equations without a cosmological constant despite the fact that the internal space may be a group manifold (cf the discussion following (3.34)) we learn that all geodesic Killing vector fields (like the $\tilde{e}_5$ and Y_a) necessarily have to satisfy

$$\langle\nabla^A X, \nabla_A X\rangle=0 \tag{4.12}$$

as a consequence of the Bochner-Yano formula (4.6). We will now turn to show that (4.12) is precisely the constraint equation (3.21) or (3.35) obtained from the field equations if (4.12) is evaluated for these special manifolds (this is not unreasonable, since (4.12) was obtained from purely geometrical considerations whereas the Einstein field equations determine the geometry).

5. The meaning of the constraints

Let us again start by looking at the simple five-dimensional model with $D=1$ and $G=U(1)\simeq S^1$ (not semisimple): the Killing vector of the metric (3.1) was $\tilde{e}_5=\partial_5$ and, with the aid of (3.6)-(3.9) we get

$$\begin{aligned}&\nabla^5\tilde{e}_5=0\\&\nabla^\mu\tilde{e}_5=\tfrac{1}{2}F^{\mu\nu}\tilde{e}_\nu.\end{aligned} \tag{5.1}$$

Then—under the assumption that (3.1) is a solution of the five-dimensional Einstein equations—(4.12) tells us that

$$\begin{aligned}0&=\langle\nabla_A\tilde{e}_5, \nabla^A\tilde{e}_5\rangle\\&=\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{5.2}$$

which is exactly the constraint (3.21). In the Riemannian case we can even deduce the stronger statement $F=0$ as a consequence of $\nabla_A\tilde{e}_5=0$ which is in this case equivalent to the constraint $F\wedge * F=0$. Thus in the Riemannian case the Einstein equations tell us that the horizontal distribution spanned by the $\tilde{e}_\mu$ is integrable.

In the case $D>1$ and G semisimple and compact the Killing vectors of the metric

$$g=g_{\mu\nu}\,\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu+g_{ab}(\hat{\theta}^a-A^a)\otimes(\hat{\theta}^b-A^b) \tag{3.22}$$

where the left-invariant vector fields Y_a satisfy the Lie algebra relation

$$[Y_a, Y_b] = C^c{}_{ab} Y_c. \tag{2.14}$$

Using the connection forms (3.25)-(3.27) we obtain

$$\nabla_\nu Y_a = \tfrac{1}{2} F_b{}^\mu{}_\nu \mathrm{D}^b{}_a \tilde{e}_\mu \tag{5.3}$$

$$\langle \nabla_\nu Y_a, \nabla^\nu Y_a \rangle = \tfrac{1}{4} F_{b\mu\nu} F_c^{\mu\nu} \mathrm{D}^b{}_a \mathrm{D}^c{}_a \tag{5.4}$$

$$\nabla_e Y_a = \tfrac{1}{2} C^d{}_{eb} \mathrm{D}^b{}_a \hat{Y}_d \tag{5.5}$$

$$\langle \nabla_e Y_a, \nabla^e Y_a \rangle = \tfrac{1}{4} g_{bc} \mathrm{D}^b{}_a \mathrm{D}^c{}_a. \tag{5.6}$$

Therefore, (4.12) leads to

$$0 = \langle \nabla_A Y_a, \nabla^A Y_a \rangle = \tfrac{1}{4}(F_{b\mu\nu} F_c^{\mu\nu} + g_{bc}) \mathrm{D}^b{}_a \mathrm{D}^c{}_a \tag{5.7}$$

which is again precisely the constraint (3.25). This proves the claim that the occurrence of these constraints in Kaluza–Klein theories is a consequence of the presence of Killing vectors of the total metric in the traditional ansatz.

6. Komar currents of Kaluza–Klein Killing vectors

In [18] Benn has shown that Killing vectors lead to the existence of closed 2-forms which he called—in generalisation of Komar's conserved quantities [16]—Komar 2-forms.

More precisely, given a Killing vector field X of g, g satisfying the Einstein field equations, we have

$$\mathrm{d} * \mathrm{d}g(X) = 0 \tag{6.1}$$

where $g(X)$ is the metric dual 1-form of X defined by

$$i_Y g(X) = \langle X, Y \rangle \qquad \forall Y \tag{6.2}$$

and $* \,\mathrm{d}g(X)$ is the Komar form.

In the case of $n = D+4$ dimensions, Killing vectors yield closed $(D+2)$-forms, as we shall explicitly check below. In the Abelian $(D=1)$ case we have

$$g(\tilde{e}_5) = g(\partial_5) = \mathrm{d}x^5 + A$$

$$\overset{5}{*}\, \mathrm{d}g(\tilde{e}_5) = \overset{5}{*}\, F = \overset{4}{*}\, F \wedge e^5 \tag{6.3}$$

$$\mathrm{d} \overset{5}{*}\, \mathrm{d}g(\tilde{e}_5) = \mathrm{d} \overset{4}{*}\, F \wedge e^5 + F \wedge \overset{4}{*}\, F = 0 \tag{6.4}$$

as a consequence of the field equations (3.19) and (3.21).

The associated de Rham period

$$Q = \int \overset{5}{*}\, F \tag{6.5}$$

(where $\int$ stands for evaluation on non-trivial spacelike 3-cycles of $M_4 \times S^1$) is proportional to the electric charge and charge conservation is thus a consequence of five-dimensional gravity.

The case $D > 1$ is not that trivial. By taking into consideration that Y_a is left-invariant, whereas the metric g (3.22) is defined in terms of right-invariant forms, one finds

$$g(Y_a) = -\mathrm{D}_{ba} T^b. \tag{6.6}$$

Using (2.9) and (2.21) $\mathrm{d}g(Y_a)$ is found to be

$$\mathrm{d}g(Y_a) = \mathrm{D}_{da} F^d + \tfrac{1}{2} C^d{}_{bc} T^c \wedge T^b \mathrm{D}_{da}. \tag{6.7}$$

Therefore

$$\begin{aligned} \mathrm{d} \overset{n}{*} \mathrm{d}g(Y_a) = {} & \mathrm{dD}_{ba} \wedge \overset{n}{*} F^b + \mathrm{D}_{ba} \mathrm{d} \overset{n}{*} F^d \\ & + \tfrac{1}{2} C^e{}_{bc}\, \mathrm{dD}_{ea} \wedge \overset{n}{*} T^c \wedge T^b + \tfrac{1}{2} C^e{}_{bc} \mathrm{D}_{ea}\, \mathrm{d} \overset{n}{*} T^c \wedge T^b \end{aligned} \tag{6.8}$$

and splitting $\overset{n}{*}$ into $\overset{4}{*}$ and $\overset{D}{*}$, the first two terms of (6.8) turn out to be

$$\mathrm{D}_{da}[(\mathrm{d} \overset{4}{*} F^d + (A \wedge \overset{4}{*} F)^d) \wedge \overset{D}{*} 1 + \overset{4}{*} F^d \wedge \mathrm{d} \overset{D}{*} 1].$$

$\mathrm{d} \overset{D}{*} 1$ is most readily computed by writing it as

$$\mathrm{d} \overset{D}{*} 1 = -\mathrm{d} \overset{n}{*} \overset{4}{*} 1 = -F_a \wedge \overset{D}{*} T^a. \tag{6.9}$$

Finally the last two terms of (6.8) yield $-\tfrac{1}{2}\mathrm{D}_{ea} * T^e$, because

$$\mathrm{d} \overset{n}{*} T^c \wedge T^b = -C^{cb}{}_a \overset{n}{*} T^a \tag{6.10}$$

and the resulting expression for $\mathrm{d} \overset{n}{*} \mathrm{d}g(Y_a)$ is

$$\mathrm{d} \overset{n}{*} \mathrm{d}g(Y_a) = \mathrm{D}_{ca}(\mathrm{d} \overset{4}{*} F^c + (A \wedge \overset{4}{*} F)^c) \wedge \overset{D}{*} 1 - \mathrm{D}^d{}_a F_b \wedge \overset{4}{*} F_d \wedge \overset{D}{*} T^b - \tfrac{1}{2} D_{ea} \overset{n}{*} T^e. \tag{6.11}$$

The first term vanishes as a consequence of the YM field equations, while the other terms cancel as a consequence of the constraint (3.35) which may also be written as

$$F_a \wedge \overset{4}{*} F_b = -\tfrac{1}{2} g_{ab} \overset{4}{*} 1.$$

Thus

$$\mathrm{d} \overset{n}{*} \mathrm{d}g(Y_a) = 0$$

if g satisfies the vacuum Einstein field equations in $D+4$ dimensions.

7. Generalised ADM charges in Kaluza–Klein theories

In the preceding section we introduced conserved currents associated with Killing vectors of the $(4+D)$-dimensional spacetime manifold. We shall now investigate those quantities corresponding to the familiar ADM energy and momentum. In a suitably [23] asymptotically flat spacetime these are usually defined as boundary integrals over 2-spheres at spacelike infinity, after a canonical $(3+1)$ decomposition has been performed with respect to a particular spacelike hypersurface Σ. Since performing a canonical decomposition of the Kaluza-Klein metrics is quite laborious and somewhat lengthy, we shall rather isolate a set of conserved quantities directly from the Einstein equations in the $(D+4)$-dimensional (instead of a $[(D+3)+1]$-dimensional) picture. Those quantities are related to generalised Landau-Lifshitz [24] pseudotensors and we shall—to gain some confidence in these coordinate dependent expressions—show that in the four-dimensional case their surface integrals coincide with the familiar ADM energy-momentum expressions, if evaluated in an asymptotically Minkowskian frame.

The Einstein equations

$$\Omega_{AB} \wedge * e^{ABC} = 0 \tag{7.1}$$

may—by splitting off an exact term from the left-hand side—be written in 'Maxwell form' as [25]

$$\mathrm{d} * f^{c} = - * J^{c} \tag{7.2}$$

with

$$*f^{c} = -\tfrac{1}{2}(\omega_{AB} \wedge * e^{ABC}) \tag{7.3}$$

and

$$*J^{c} = \tfrac{1}{2}\omega_{AB} \wedge (\omega^{C}{}_{D} \wedge * e^{ABD} - \omega^{B}{}_{D} \wedge * e^{ACD}). \tag{7.4}$$

In a coordinate basis J^{c} coincides with the Landau-Lifshitz pseudotensor and (7.2) implies the conservation law

$$\mathrm{d} * J^{c} = 0. \tag{7.5}$$

Since

$$\int_{\Sigma} * J^{c} = -\int_{\partial\Sigma} * f^{c} \tag{7.6}$$

we are interested in the asymptotic form of $*f^{c}$. In four dimensions we obtain

$$\begin{aligned} \overset{4}{*} f^{0} &= -\tfrac{1}{2}\omega_{\mu\nu} \wedge \overset{4}{*} e^{\mu\nu 0} \qquad && \mu, \nu = 0, 1, 2, 3 \\ &= -\tfrac{1}{2}\omega_{ij} \wedge \overset{4}{*} e^{ij0} && i, j = 1, 2, 3 \\ &= \tfrac{1}{2}\omega_{ij} \wedge \overset{3}{*} e^{ij}. \end{aligned}$$

In a cartesian frame—which is valid asymptotically—this reduces to

$$\begin{aligned} \overset{4}{*} f^{0} &= \tfrac{1}{2}\Gamma_{ijk}\, \mathrm{d}x^{k} \wedge \overset{3}{*}\, \mathrm{d}x^{i} \wedge \mathrm{d}x^{j} \\ &= \tfrac{1}{2}\Gamma_{ijk}(g^{ki} * \mathrm{d}x^{j} - g^{kj} * \mathrm{d}x^{i}) \\ &= \tfrac{1}{2}(g^{ik}g_{ik,j} - g_{jk,}{}^{k}) * \mathrm{d}x^{j} \end{aligned} \tag{7.7}$$

which is just the ADM energy integrand [17]. Likewise $*f^{i}$ may be seen to lead to the ADM momentum and we are therefore tempted to evaluate $*f^{A}$ for our Kaluza-Klein metrics as well.

In the case of a five-dimensional manifold of the form $M_5 = M_4 \times S^1$, we replace Σ by $\Sigma \times S^1$ where Σ is an asymptotically flat spacelike hypersurface of M_4 and integrate $*f^{A}$ over $\partial(\Sigma \times S^1) = \partial\Sigma \times S^1$ For f^0 we find (using the connection forms (3.6)-(3.9))

$$\begin{aligned} \int_{\partial\Sigma\times S^1} \overset{5}{*} f^{0} &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} \omega_{BC} \wedge \overset{5}{*} e^{BC0} \\ &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} \omega_{\mu\nu} \wedge \overset{5}{*} e^{\mu\nu 0} - \int_{\partial\Sigma\times S^1} \omega_{\mu 5} \wedge \overset{5}{*} e^{\mu 50} \\ &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} \omega_{ij} \wedge \overset{5}{*} e^{ij0} \\ &= 2\pi R E_{\mathrm{ADM}} \end{aligned} \tag{7.8}$$

where E_{ADM} is the integral of (7.7) over Σ and R is the 'radius' of S^1. Thus (7.8) coincides—up to a numerical factor—with the four-dimensional value for the energy, which is not surprising since the metric g implicitly contains—as a solution of the five-dimensional field equations—all the information about the energy of the electromagnetic field.

The only reasonable candidate for the fifth component of the ADM momentum is probably the electric charge and this is indeed true because

$$\begin{aligned}\int_{\partial\Sigma\times S^1} \overset{5}{*} f^5 &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} \omega_{\mu\nu}\wedge \overset{5}{*}\, e^{\mu\nu 5}\\ &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} \omega_{\mu\nu}\wedge \overset{4}{*}\, e^{\mu\nu}\\ &= -\tfrac{1}{2}\int_{\partial\Sigma\times S^1} {}^{(4)}\omega_{\mu\nu}\wedge \overset{4}{*}\, e^{\mu\nu} + \tfrac{1}{4}\int_{\partial\Sigma\times S^1} F_{\mu\nu}e^5\wedge \overset{4}{*}\, e^{\mu\nu}\\ &= \tfrac{1}{2}\int_{S^1} \mathrm{d}x^5 \int_{\partial\Sigma} \overset{4}{*} F = R\pi Q.\end{aligned} \tag{7.9}$$

This result should be compared with (6.5) where we obtained the electric charge as the de Rham period associated with the Komar form of the Killing vector $\tilde{e}_5 = \partial_5$.

In the YM case we proceed analogously and obtain for the integrand $*f^d$ of the 'internal' ADM momentum

$$\begin{aligned}\omega_{AB}\wedge \overset{n}{*}\, e^{ABd}\big|_{\partial\Sigma\times G} &= T^a \wedge \overset{n}{*}\,(F_a\wedge T^d)\\ &= \overset{n}{*}\, F^d\\ &= \overset{4}{*}\, F^d \wedge \overset{D}{*}\, 1.\end{aligned} \tag{7.10}$$

Therefore the integral will yield the YM charge Q^d including its own non-linear contribution

$$\overset{4}{*}\, J^a_{\mathrm{YM}} := (A\wedge \overset{4}{*}\, F)^a \tag{7.11}$$

whereas the integral of the Komar form $*\,\mathrm{d}g(Y_a)$ measures only the charge of other fields possibly coupled to the YM field.

Assume that F^a satisfies

$$\mathrm{d}\overset{4}{*}\, F^a + (A\wedge \overset{4}{*}\, F)^a = -\overset{4}{*}\, J^a_{\mathrm{ext}} \tag{7.12}$$

where J_{ext} is some external current. Then we obtain

$$\int_{\partial\Sigma\times G} \overset{n}{*} f^a = -\int_{\Sigma\times G} \overset{n}{*}\,(J^a_{\mathrm{ext}} + J^a_{\mathrm{YM}}) \tag{7.13}$$

as opposed to (cf (6.7)-(6.9))

$$\begin{aligned}\int_{\partial\Sigma\times G} \overset{n}{*}\, \mathrm{d}g(Y_a) &= \int_{\partial\Sigma\times G} \mathrm{D}_{da}(\overset{n}{*}\, F^d + \tfrac{1}{2}C^d{}_{bc}\overset{n}{*}\, T^c\wedge T^b)\\ &= \int_{\partial\Sigma\times G} \mathrm{D}_{ba}\overset{n}{*}\, F^b\\ &= \int_{\Sigma\times G} \mathrm{D}_{ba}(\mathrm{d}\overset{4}{*}\, F^b + (A\wedge \overset{4}{*}\, F)^b)\wedge \overset{D}{*}\, 1 - \int_{\Sigma\times G} \mathrm{D}^c{}_a F_b \wedge \overset{4}{*}\, F_c \wedge \overset{D}{*}\, T^b\\ &= -\int_{\Sigma\times G} \mathrm{D}_{ba}\overset{4}{*}\, J^b_{\mathrm{ext}}\wedge \overset{D}{*}\, 1\\ &= -\int_{\Sigma\times G} \mathrm{D}_{ba}\overset{n}{*}\, J^b_{\mathrm{ext}}.\end{aligned} \tag{7.14}$$

Thus we conclude that whereas the ADM energy-momentum is preserved in the presence of external fields, the Komar quantities are not, since the Komar form $* \mathrm{d}g(Y_a)$ fails to be closed in that case (or—more physically—fails to take into account an exchange of YM charge between the YM field and the external fields coupled to it).

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