Solutions to Assignments 05

1. It is useful to first recall how this works in the case of classical mechanics (i.e. a 0+1 dimensional "field theory"). Consider a Lagrangian $L(q,\dot{q};t)$ that is a total time-derivative, i.e.

$$L(q,\dot{q};t) = \frac{d}{dt}F(q;t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q(t)}\dot{q}(t) \quad . \tag{1}$$

Then one has

$$\frac{\partial L}{\partial q(t)} = \frac{\partial^2 F}{\partial q(t)\partial t} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{2}$$

and

$$\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial F}{\partial q(t)} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial^2 F}{\partial t \partial q(t)} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{3}$$

Therefore one has

$$\frac{\partial L}{\partial q(t)} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} \qquad \text{identically} \tag{4}$$

Now we consider the field theory case. We define

$$\begin{aligned} L := \frac{d}{dx^\alpha}W^\alpha(\phi;x) &= \frac{\partial W^\alpha}{\partial x^\alpha} + \frac{\partial W^\alpha}{\partial \phi(x)}\frac{\partial \phi(x)}{\partial x^\alpha} \\ &= \partial_\alpha W^\alpha + \partial_\phi W^\alpha \partial_\alpha \phi \end{aligned} \tag{5}$$

to compute the Euler-Lagrange equations for it. One gets

$$\frac{\partial L}{\partial \phi} = \partial_\phi \partial_\alpha W^\alpha + \partial_\phi^2 W^\alpha \partial_\alpha \phi \tag{6}$$

$$\begin{aligned} \frac{d}{dx^\beta}\frac{\partial L}{\partial(\partial_\beta \phi)} &= \frac{d}{dx^\beta}\left(\partial_\phi W^\alpha \delta_\alpha^{\ \beta}\right) \\ &= \partial_\beta \partial_\phi W^\beta + \partial_\phi^2 W^\beta \partial_\beta \phi \; . \end{aligned} \tag{7}$$

Thus (since partial derivatives commute) the Euler-Lagrange equations are satisfied identically.

Remark: One can also show the converse: if a Lagrangian L gives rise to Euler-Lagrange equations that are identically satisfied then (locally) the Lagrangian is a total derivative. The proof is simple. Assume that $L(q,\dot{q};t)$ satisfies

$$\frac{\partial L}{\partial q} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{8}$$

identically. The left-hand side does evidently not depend on the acceleration $\ddot{q}$. The right-hand side, on the other hand, will in general depend on $\ddot{q}$ - unless L is at most linear in $\dot{q}$. Thus a necessary condition for L to give rise to identically satisfied Euler-Lagrange equations is that it is of the form

$$L(q,\dot{q};t) = L^0(q;t) + L^1(q;t)\dot{q} \quad . \tag{9}$$

Therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}L^1 = \frac{\partial L^1}{\partial t} + \frac{\partial L^1}{\partial q}\dot{q} \tag{10}$$

and

$$\frac{\partial L}{\partial q} = \frac{\partial L^0}{\partial q} + \frac{\partial L^1}{\partial q}\dot{q} \quad . \tag{11}$$

Noting that the 2nd terms of the previous two equations are equal, the Euler-Lagrange equations thus reduce to the condition

$$\frac{\partial L^1}{\partial t} = \frac{\partial L^0}{\partial q} \quad . \tag{12}$$

This means that locally there is a function $F(q;t)$ such that

$$L^0 = \partial_t F \quad , \quad L^1 = \partial_q F \quad , \tag{13}$$

and therefore

$$L = L^0 + L^1\dot{q} = \partial_t F + \partial_q F \dot{q} = \frac{d}{dt}F \quad , \tag{14}$$

as was to be shown. (Proof in the field theory case is analogous)

2. Complex Scalar Field I: Action and Equations of Motion

The action is

$$\begin{aligned} S[\Phi] &= \int d^4x \left(-\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi^* - W(\Phi,\Phi^*)\right) \\ &= \int d^4x \left(-\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_1 - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi_2\partial_\beta\phi_2 - V(\phi_1,\phi_2)\right) \end{aligned} \tag{15}$$

(a) Varying ϕ_1 in the 2nd line, while keeping ϕ_2 fixed, one finds

$$\delta S = \int d^4x \left(-\eta^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\delta\phi_1 - (\partial V/\partial\phi_1)\delta\phi_1\right) \quad . \tag{16}$$

Integrating by parts the first term, and dropping the boundary term, one finds the Euler-Lagrange equation $\Box\phi_1 = \partial V/\partial\phi_1$. Analogous for ϕ_2.

(b) Using

$$\begin{aligned} \partial V/\partial\phi_1 &= (\partial W/\partial\Phi)(\partial\Phi/\partial\phi_1) + (\partial W/\partial\Phi^*)(\partial\Phi^*/\partial\phi_1) = (\partial W/\partial\Phi) + (\partial W/\partial\Phi^*) \\ \partial V/\partial\phi_2 &= (\partial W/\partial\Phi)(\partial\Phi/\partial\phi_2) + (\partial W/\partial\Phi^*)(\partial\Phi^*/\partial\phi_2) = i(\partial W/\partial\Phi) - i(\partial W/\partial\Phi^*) \end{aligned} \tag{17}$$

one finds

$$\begin{aligned} \Box\Phi &= \Box\phi_1 + i\Box\phi_2 = \partial V/\partial\phi_1 + i\partial V/\partial\phi_2 = 2\partial W/\partial\Phi^* \\ \Box\Phi^* &= \Box\phi_1 - i\Box\phi_2 = \partial V/\partial\phi_1 - i\partial V/\partial\phi_2 = 2\partial W/\partial\Phi \end{aligned} \tag{18}$$

(c) Varying only Φ^* in the first line of the action, while keeping Φ fixed, one finds

$$\delta S = \int d^4x \left(-\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\delta\Phi^* - (\partial W/\partial\Phi^*)\delta\Phi^*\right) \quad . \tag{19}$$

Integrating by parts the 1st term, one obtains $(1/2)\Box\Phi$, and thus the correct Euler-Lagrange equation for Φ (analogously for $\Phi \leftrightarrow \Phi^*$).

3. Complex Scalar Field II: Phase Invariance and Noether-Theorem

(a) If the potential is a function of $\Phi^*\Phi$, both the potential and the derivative terms of the Lagrangian are obviously invariant under

$$\Phi(x) \to \mathrm{e}^{i\theta}\Phi(x) \quad , \quad \Phi^*(x) \to \mathrm{e}^{-i\theta}\Phi^*(x) \tag{20}$$

for constant θ, since in this case the derivatives transform the same way, i.e.

$$\partial_\alpha\Phi(x) \to \mathrm{e}^{i\theta}\partial_\alpha\Phi(x) \quad , \quad \partial_\alpha\Phi^*(x) \to \mathrm{e}^{-i\theta}\partial_\alpha\Phi^*(x) \tag{21}$$

(b) Infinitesimally, one has

$$\Delta\Phi = i\theta\Phi \quad , \quad \Delta\Phi^* = -i\theta\Phi^* \ , \tag{22}$$

and therefore the corresponding Noether current is

$$J^\alpha_\Delta = \frac{\partial L}{\partial(\partial_\alpha\Phi)}\Delta\Phi + \frac{\partial L}{\partial(\partial_\alpha\Phi^*)}\Delta\Phi^* = -(i\theta/2)(\Phi\partial^\alpha\Phi^* - \Phi^*\partial^\alpha\Phi) \tag{23}$$

where (as usual) $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$. Calculating its divergence, one finds (ignoring the irrelevant constant prefactor, and using the equations of motion)

$$\begin{aligned}\partial_\alpha(\Phi\partial^\alpha\Phi^* - \Phi^*\partial^\alpha\Phi) &= \partial_\alpha\Phi\partial^\alpha\Phi^* + \Phi\Box\Phi^* - \partial_\alpha\Phi^*\partial^\alpha\Phi - \Phi^*\Box\Phi \\ &= \Phi\Box\Phi^* - \Phi^*\Box\Phi = 2(\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^*)\end{aligned} \tag{24}$$

This is not (and should not be) zero in general, but it is zero precisely when $W = W(\Phi^*\Phi)$. Indeed, in that case one has

$$\partial W(\Phi^*\Phi)/\partial\Phi = W'(\Phi^*\Phi)\Phi^* \quad , \quad \partial W(\Phi^*\Phi)/\partial\Phi^* = W'(\Phi^*\Phi)\Phi \ , \tag{25}$$

and therefore

$$\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^* = W'(\Phi^*\Phi)\,(\Phi\Phi^* - \Phi^*\Phi) = 0 \ . \tag{26}$$

4. Complex Scalar Field III: Gauge Invariance and Minimal Coupling

(a) Under

$$\Phi(x) \to \mathrm{e}^{i\theta(x)}\Phi(x) \quad , \quad \Phi^*(x) \to \mathrm{e}^{-i\theta(x)}\Phi^*(x) \quad , \quad A_\alpha(x) \to A_\alpha(x) + \partial_\alpha\theta(x) \tag{27}$$

the partial derivative transforms as

$$\partial_\alpha\Phi \to \partial_\alpha(\mathrm{e}^{i\theta}\Phi) = \mathrm{e}^{i\theta}(\partial_\alpha\Phi + i(\partial_\alpha\theta)\Phi) \tag{28}$$

Therefore the covariant derivative

$$D_\alpha\Phi = \partial_\alpha\Phi - iA_\alpha\Phi \quad , \quad D_\alpha\Phi^* = \partial_\alpha\Phi^* + iA_\alpha\Phi^* \ . \tag{29}$$

transforms as

$$\begin{aligned} D_\alpha \Phi \to & \, \mathrm{e}^{i\theta}(\partial_\alpha \Phi + i(\partial_\alpha \theta)\Phi) - i\mathrm{e}^{i\theta} A_\alpha \Phi - i\mathrm{e}^{i\theta}(\partial_\alpha \theta)\Phi \\ & = \mathrm{e}^{i\theta}(\partial_\alpha \Phi - i A_\alpha \Phi) = \mathrm{e}^{i\theta} D_\alpha \Phi \end{aligned} \tag{30}$$

Likewise

$$D_\alpha \Phi^* \to \mathrm{e}^{-i\theta} D_\alpha \Phi^* \quad . \tag{31}$$

(b) It is now obvious that the action

$$S[\Phi, A] = \int d^4x \left(-\tfrac{1}{2}\eta^{\alpha\beta} D_\alpha \Phi D_\beta \Phi^* - W(\Phi\Phi^*) \right) \tag{32}$$

ist gauge invariant.

(c) The action is

$$S = S_{\mathrm{Maxwell}}[A] + S[\Phi, A] = \int d^4x (-\tfrac{1}{4}F^2) + S[\Phi, A] \quad . \tag{33}$$

The equations of motion for Φ and Φ^* are simply the covariant versions of the equations of motion from Exercise 2, namely

$$D^\alpha D_\alpha \Phi = 2\partial W/\partial \Phi^* \quad , \quad D^\alpha D_\alpha \Phi^* = 2\partial W/\partial \Phi \quad . \tag{34}$$

Variation with respect to A leads to

$$\delta S = \int d^4x \left(\partial_\alpha F^{\alpha\beta} + J^\beta \right) \delta A_\beta \tag{35}$$

where

$$J^\beta = (i/2) \left(\Phi D^\beta \Phi^* - \Phi^* D^\beta \Phi \right) \tag{36}$$

The equations of motion $\partial_\alpha F^{\alpha\beta} + J^\beta = 0$ imply (and therefore require) that $\partial_\beta J^\beta = 0$. Let us show that this equation is satisfied as a consequence of the equations of motion for Φ.

First of all, we have

$$\partial_\beta (\Phi D^\beta \Phi^*) = \partial_\beta \Phi D^\beta \Phi^* + \Phi \partial_\beta D^\beta \Phi^* \quad . \tag{37}$$

Adding and subtracting $+iA_\beta \Phi$, we can write this as

$$\partial_\beta (\Phi D^\beta \Phi^*) = D_\beta \Phi D^\beta \Phi^* + \Phi D_\beta D^\beta \Phi^* \quad . \tag{38}$$

Since the first term is invariant under the exchange $\Phi \leftrightarrow \Phi^*$, one finds

$$\partial_\beta \left(\Phi D^\beta \Phi^* - \Phi^* D^\beta \Phi \right) = \Phi D_\beta D^\beta \Phi^* - \Phi^* D_\beta D^\beta \Phi \tag{39}$$

Note that this is just the covariant version of the divergence of the Noether current in Exercise 3, and the remaining step in the proof that this vanishes for a solution to the equations of motion is now identical to that in Exercise 3.