Solutions to Assignments 05

1. Noether Current and Noether Energy-Momentum Tensor for Field Theories

   (a) An infinitesimal rotation in the Φ_1, Φ_2 field space reads

$$\begin{aligned} \Delta\Phi_1 &= \gamma\Phi_2 \\ \Delta\Phi_2 &= -\gamma\Phi_1 \, . \end{aligned} \tag{1}$$

   Under such a transformation, the potential $V(\Phi_1^2 + \Phi_2^2)$ is trivially invariant and the kinetic term can also be seen to be invariant,

$$\Delta S_{\text{kin}} = -\frac{1}{2}\int d^4x \, 2\left(\partial_\alpha\Phi_1\partial^\alpha\gamma\Phi_2 - \partial_\alpha\gamma\Phi_1\partial^\alpha\Phi_2\right) = 0 \quad \Rightarrow \quad \Delta S = 0 \, . \tag{2}$$

   The current in given by

$$J^\alpha_\Delta = \frac{\partial L}{\partial(\partial_\alpha\Phi_a)}\Delta\Phi_a = -\gamma\left(\Phi_1\partial^\alpha\Phi_2 - \Phi_2\partial^\alpha\Phi_1\right) \tag{3}$$

   and for a solution Φ_a to the equations of motions

$$\Box\Phi_a = \frac{\partial V}{\partial\Phi_a} = 2V'(\Phi_1^2 + \Phi_2^2)\Phi_a \tag{4}$$

   one can explicitly check that

$$\begin{aligned} \partial_\alpha J^\alpha_\Delta &= -\gamma\left(\partial_\alpha\Phi_1\partial^\alpha\Phi_2 + \Phi_1\Box\Phi_2 - \partial_\alpha\Phi_2\partial^\alpha\Phi_1 - \Phi_2\Box\Phi_1\right) \\ &= -2\gamma\left(\Phi_1\Phi_2 - \Phi_2\Phi_1\right)V'(\Phi_1^2 + \Phi_2^2) \\ &= 0 \, . \end{aligned} \tag{5}$$

   (b) The Energy-Momentum Tensor, or Stress-Energy tensor (also der Spannungs-Energie Tensor, ein alternativer englischer Begriff für den Energie-Impuls Tensor) is given by

$$\theta^\alpha{}_\beta = -\partial^\alpha\Phi_a\partial_\beta\Phi_a + \delta^\alpha{}_\beta\left(\frac{1}{2}\partial^\gamma\Phi_a\partial_\gamma\Phi_a + V(\Phi_a)\right) \, . \tag{6}$$

   It is conserved for Φ_a a solution to the equations of motion $\Box\Phi_a = \frac{\partial V}{\partial\Phi_a}$:

$$\begin{aligned} \partial_\alpha\theta^\alpha{}_\beta &= -\Box\Phi_a\partial_\beta\Phi_a - \partial^\alpha\Phi_a\partial_\alpha\partial_\beta\Phi_a + \partial^\gamma\Phi_a\partial_\beta\partial_\gamma\Phi_a + \partial_\beta V(\Phi_a) \\ &= -\frac{\partial V}{\partial\Phi_a}\partial_\beta\Phi_a + \partial_\beta V(\Phi_a) \\ &= 0 \, . \end{aligned} \tag{7}$$

2. The Maxwell Energy-Momentum Tensor

(a) The trace of $T^\alpha_{\ \beta}$ is

$$T^\alpha_{\ \alpha} = -F^{\alpha\gamma}F_{\alpha\gamma} + \tfrac{1}{4}\delta^\alpha_{\ \alpha}\, F_{\gamma\delta}F^{\gamma\delta} = -F^{\alpha\gamma}F_{\alpha\gamma} + F_{\gamma\delta}F^{\gamma\delta} = 0\ . \tag{8}$$

Then one shows that $T_{\alpha\beta}$ is symmetric computing

$$T_{\alpha\beta} = \eta_{\alpha\gamma}T^\gamma_{\ \beta} = -F_\alpha^{\ \rho}F_{\beta\rho} + \tfrac{1}{4}\eta_{\alpha\beta}F_{\rho\delta}F^{\rho\delta}\ , \tag{9}$$

and using the fact that $\eta_{\alpha\beta}$ and $F_\alpha^{\ \rho}F_{\beta\rho} = F_{\alpha\rho}F_\beta^{\ \rho} = F_\beta^{\ \rho}F_{\alpha\rho}$ are symmetric.

(b) The component $T^0_{\ 0}$ is

$$\begin{aligned} T^0_{\ 0} &= -F^{0i}F_{0i} + \tfrac{1}{4}F_{\gamma\delta}F^{\gamma\delta} \\ &= -(c^{-1}E^i)(-c^{-1}E_i) + \tfrac{1}{2}(B^2 - c^{-2}E^2) = \frac{1}{2}\left(E^2/c^2 + B^2\right)\ , \end{aligned} \tag{10}$$

where $\frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}$ has been computed in assignments 03 (exercise 4). The component $T^k_{\ 0}$ is

$$\begin{aligned} T^k_{\ 0} &= -F^{k\gamma}F_{0\gamma} = -F^{kj}F_{0j} \\ &= \epsilon_{kji}B_i c^{-1}E_j = c^{-1}\epsilon_{kji}E_jB_i = S_k/c = S^k/c \end{aligned} \tag{11}$$

(c) One easily computes (Lines 1+2)

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= -F^\beta_{\ \gamma}\partial_\alpha F^{\alpha\gamma} - F^{\alpha\gamma}\partial_\alpha F^\beta_{\ \gamma} + \tfrac{1}{2}\eta^{\alpha\beta}F^{\gamma\delta}\partial_\alpha F_{\gamma\delta} \\ &= J_\gamma F^{\beta\gamma} + \eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} + \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\partial_\lambda \end{aligned} \tag{12}$$

where the Maxwell equation $\partial_\alpha F^{\alpha\gamma} = -J^\gamma$ was used in the 1st term, and some indices have been relabelled in the 2nd term to make it more similar to the 3rd term. Now we can use the antisymmetry of $F^{\gamma\delta}$ to rewrite the 2nd term as (Line 3)

$$\eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} = \tfrac{1}{2}\eta^{\beta\lambda}F^{\gamma\delta}(\partial_\delta F_{\lambda\gamma} - \partial_\gamma F_{\lambda\delta})\ . \tag{13}$$

Plugging this into the previous result and using the homogeneous Maxwell equations, one finds (Line 4)

$$\partial_\alpha T^{\alpha\beta} = -J_\gamma F^{\gamma\beta} + \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\left(\partial_\lambda F_{\gamma\delta} + \partial_\delta F_{\lambda\gamma} + \partial_\gamma F_{\delta\lambda}\right) = -J_\gamma F^{\gamma\beta}\ . \tag{14}$$