KFT Solutions 03

1. Inhomogeneous Maxwell-Equations and Potentials

(a) Under the gauge transformation $A_\beta \to A_\beta + \partial_\beta \Psi$, $F_{\alpha\beta}$ transforms as

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha \to \partial_\alpha A_\beta + \partial_\alpha \partial_\beta \Psi - \partial_\beta A_\alpha - \partial_\beta \partial_\alpha \Psi = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1}$$

and therefore $F_{\alpha\beta}$ is gauge-invariant.

(b) With $A_\alpha = (-\phi/c, \vec{A})$ one has

$$\begin{aligned} F_{0k} &= -F_{k0} = \partial_0 A_k - \partial_k A_0 = c^{-1}(\partial_t A_k + \partial_k \phi) = -E_k/c \\ F_{ik} &= \partial_i A_k - \partial_k A_i = \epsilon_{ik\ell} B_\ell \quad (F_{12} = B_3 \quad \text{etc.}) \end{aligned} \tag{2}$$

and therefore, with $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta} F_{\gamma\delta}$,

$$F^{0k} = -F^{k0} = -F_{0k} = E_k/c \quad , \quad F^{ik} = F_{ik} = \epsilon_{ik\ell} B_\ell \ . \tag{3}$$

(c) Thus, with $J^\alpha = (\rho c, \vec{J})$ one has

$$\partial_\alpha F^{\alpha 0} = \partial_k F^{k0} = -c^{-1} \vec{\nabla}.\vec{E} = -\rho/(\epsilon_0 c) = -\mu_0 c \rho = -\mu_0 J^0 \tag{4}$$

and

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = c^{-2} \partial_t E_1 - \partial_2 B_3 + \partial_3 B_2 \\ &= -(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E})_1 = -\mu_0 J_1 = -\mu_0 J^1 \end{aligned} \tag{5}$$

(and likewise for the 2- and 3-components).

(d) One has

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \Box A^\beta - \partial^\beta \partial_\alpha A^\alpha = -\mu_0 J^\beta \tag{6}$$

and therefore $\Box A_\beta - \partial_\beta \partial_\alpha A^\alpha = -\mu_0 J_\beta$.

(e) One has

$$\Box(A_\beta + \partial_\beta \Psi) - \partial_\beta \partial_\alpha (A^\alpha + \partial^\alpha \Psi) = \Box A_\beta + \partial_\beta \Box \Psi - \partial_\beta \partial_\alpha A^\alpha - \partial_\beta \Box \Psi \tag{7}$$

Since the $\Box\Psi$-term cancels, the expression is gauge invariant.

(f) From $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$ one deduces $-\mu_0 \partial_\beta J^\beta = \partial_\beta \partial_\alpha F^{\alpha\beta} = 0$ because $F^{\alpha\beta} = -F^{\beta\alpha}$ is anti-symmetric while $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ is symmetric.

2. The Homogeneneous Maxwell-Equations

(a) One has $\partial_\alpha F_{\beta\gamma} = \partial_\alpha\partial_\beta A_\gamma - \partial_\alpha\partial_\gamma A_\beta$ etc. Using the fact that 2nd partial derivatives commute one deduces

$$\begin{aligned} & \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} \\ = \; & \partial_\alpha\partial_\beta A_\gamma - \partial_\gamma\partial_\alpha A_\beta + \partial_\gamma\partial_\alpha A_\beta - \partial_\beta\partial_\gamma A_\alpha + \partial_\beta\partial_\gamma A_\alpha - \partial_\alpha\partial_\beta A_\gamma = 0 \end{aligned} \tag{8}$$

(b) $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

i. Two indices equal ($\alpha = \beta$, say): $\partial_\alpha F_{\alpha\gamma} + \partial_\alpha F_{\gamma\alpha} + \partial_\gamma F_{\alpha\alpha} = 0$ is identically satisfied because $F_{\alpha\alpha} = 0$, $F_{\alpha\gamma} + F_{\gamma\alpha} = 0$.

ii. All 3 indices spatial, $(\alpha, \beta, \gamma) = (1, 2, 3)$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla}.\vec{B} = 0 \tag{9}$$

iii. One index time, the others spatial, e.g. $(\alpha, \beta, \gamma) = (0, 1, 2)$:

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1) = c^{-1}(\vec{\nabla}\times\vec{E} + \partial_t\vec{B})_3 = 0 \tag{10}$$

(and likewise for the other components).

3. The dual field strength tensor

The dual field strength tensor is defined by

$$\tilde{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \ . \tag{11}$$

With $F_{0i} = -c^{-1}E_i$, $F_{ij} = \epsilon_{ijk}B_k$ and $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ one gets for the non-zero components of $\tilde{F}^{\alpha\beta}$:

$$\tilde{F}^{0i} = \tfrac{1}{2}\epsilon^{0i\gamma\delta}F_{\gamma\delta} = \tfrac{1}{2}\epsilon^{0ijk}F_{jk} = \tfrac{1}{2}\epsilon^{0ijk}\epsilon_{jkl}B_l = -B_i \tag{12}$$

$$\tilde{F}^{ij} = \tfrac{1}{2}\epsilon^{ij\gamma\delta}F_{\gamma\delta} = \epsilon^{ij0k}F_{0k} = -c^{-1}\epsilon^{ij0k}E_k = c^{-1}\epsilon^{ijk}E_k \ . \tag{13}$$

The equation $\partial_\lambda \tilde{F}^{\lambda\delta} = 0$ can then be written as

$$\partial_\lambda \tilde{F}^{\lambda 0} = -\partial_i \tilde{F}^{0i} = \vec{\nabla}\cdot\vec{B} \tag{14}$$

$$\begin{aligned} \partial_\lambda \tilde{F}^{\lambda j} &= \partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -c^{-1}\partial_t B_j - c^{-1}\epsilon_{jik}\partial_i E_k \\ &= -\frac{1}{c}\left(\partial_t\vec{B} + \vec{\nabla}\times\vec{E}\right)_j \ , \end{aligned} \tag{15}$$

which proves the assertion.

4. Lorentz Invariants

(a)

$$
\begin{aligned}
I_1 &= \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\
&= \tfrac{1}{2}\left(-(F_{0i})^2 + (F_{ij})^2\right) = \tfrac{1}{2}\left(\vec{B}^2 - c^{-2}\vec{E}^2\right) \qquad (16)\\
I_2 &= \tfrac{1}{4}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij}\right) \\
&= \tfrac{1}{4}\left(2c^{-1}E_iB_i + \epsilon_{ijk}B_kc^{-1}\epsilon^{ijl}E_l\right) = c^{-1}\vec{E}\cdot\vec{B} \qquad (17)
\end{aligned}
$$

where one has used $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$. If $\vec{E} = 0$ in one inertial system, then $I_1 > 0$ and $I_2 = 0$ in all inertial systems, and thus $\vec{E}.\vec{B} = 0$ and $|\vec{E}| < |\vec{B}|$ in all inertial systems.

(b)

$$
8I_2 = \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = 2\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A_\beta)F_{\gamma\delta} = 2\partial_\alpha\left(\epsilon^{\alpha\beta\gamma\delta}A_\beta F_{\gamma\delta}\right) \qquad (18)
$$

because $\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = 0$ (Bianchi identity). Thus I_2 is a total derivative, $I_2 = \partial_\alpha C^\alpha$. C^α is not gauge invariant but changes by a total derivative under a gauge transformation,

$$
\epsilon^{\alpha\beta\gamma\delta}A_\beta F_{\gamma\delta} \to \epsilon^{\alpha\beta\gamma\delta}\partial_\beta\psi F_{\gamma\delta} = \partial_\beta(\epsilon^{\alpha\beta\gamma\delta}\psi F_{\gamma\delta}) \qquad (19)
$$

5. LORENTZ TRANSFORMATION OF $\vec{B}$:

Using $\bar{F}_{\alpha\beta} = \Lambda_\alpha^{\ \gamma}\Lambda_\beta^{\ \delta}F_{\gamma\delta}$ one wants to compute the transformation of $\bar{F}_{ij}$ which contains the magnetic field components. To do this we need $\Lambda_\alpha^{\ \beta}$ which is obtained from $L^\alpha_{\ \beta}$ by inverse transposition, which gives

$$
\left(\Lambda_\alpha^{\ \beta}\right) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (20)
$$

With it one computes :

$$
\begin{aligned}
\bar{F}_{ij} &= \Lambda_i^{\ \gamma}\Lambda_j^{\ \delta}F_{\gamma\delta} = \Lambda_i^{\ 0}\Lambda_j^{\ l}F_{0l} + \Lambda_i^{\ k}\Lambda_j^{\ 0}F_{k0} + \Lambda_i^{\ k}\Lambda_j^{\ l}F_{kl} \\
&= \left(\Lambda_i^{\ 0}\Lambda_j^{\ l} - \Lambda_i^{\ l}\Lambda_j^{\ 0}\right)F_{0l} + \Lambda_i^{\ k}\Lambda_j^{\ l}F_{kl} \qquad (21)
\end{aligned}
$$

such that

$$
\begin{aligned}
\bar{F}_{12} &= \sinh\alpha F_{02} + \cosh\alpha F_{12} = -c^{-1}\gamma\beta E_2 + \gamma B_3 \qquad (22)\\
\bar{F}_{23} &= F_{23} \qquad (23)\\
\bar{F}_{31} &= -\sinh\alpha F_{03} + \cosh\alpha F_{31} = c^{-1}\gamma\beta E_3 + \gamma B_2 \qquad (24)
\end{aligned}
$$

from which one can read off the transformation of the magnetic field :

$$
\begin{aligned}
\bar{B}_1 &= B_1 \\
\bar{B}_2 &= \gamma B_2 + c^{-1}\beta\gamma E_3 \\
\bar{B}_3 &= \gamma B_3 - c^{-1}\beta\gamma E_2 \qquad (25)
\end{aligned}
$$