Solutions to Assignments 01

1. Die Lorentz-Gruppe

   (a) The first claim follows from multiplicativity of the determinant (and invariance under transposition):

   $$L^T \eta L = \eta \quad \Rightarrow \quad \det(L^T \eta L) = \det(\eta) \quad \Rightarrow \quad \det(L)^2 = +1 \ . \qquad (1)$$

   The second claim follows fom writing $(L^T \eta L)_{00} = \eta_{00}$ explicitly,

   $$\begin{aligned} & \eta_{\alpha\beta} L^\alpha_0 L^\beta{}_0 \stackrel{!}{=} \eta_{00} = -1 \\ \Rightarrow \quad & \eta_{00} L^0_0 L^0{}_0 + \eta_{ik} L^i_0 L^k{}_0 = -(L^0_0)^2 + \delta_{ik} L^i_0 L^k{}_0 = -1 \\ \Rightarrow \quad & (L^0_0)^2 = 1 + \delta_{ik} L^i_0 L^k{}_0 \geq 1 \ . \end{aligned} \qquad (2)$$

   (b) It is trivial to verify that

   $$L_1^T \eta L_1 = \eta, L_2^T \eta L_2 = \eta \quad \Rightarrow \quad (L_1 L_2)^T \eta (L_1 L_2) = \eta \ . \qquad (3)$$

   Existence of an inverse L^{-1} follows from $\det L \neq 0$ (shown above). That $L \in \mathcal{L} \Rightarrow L^{-1} \in \mathcal{L}$ follows from

   $$L^T \eta L = \eta \quad \Leftrightarrow \quad \eta = (L^{-1})^T \eta L^{-1} \ . \qquad (4)$$

2. Geometrie : Analytische Minkowski-Geometrie

   (a) Because of Lorentz invariance one can choose to work in the rest frame in which $v = (a, 0, 0, 0)$. A general null vector $w = (w^0, w^1, w^2, w^3)$ is such that $(w^0)^2 = (w^1)^2 + (w^2)^2 + (w^3)^2$ and we want $v = w_1 + w_2$. With the choice $w_1^0 = w_2^0 = \frac{a}{2}$ one possibility is

   $$w_1 = (\tfrac{a}{2}, \tfrac{a}{2}, 0, 0) \quad w_2 = (\tfrac{a}{2}, -\tfrac{a}{2}, 0, 0) \ . \qquad (5)$$

   (b) Assuming $v = (a, a, 0, 0)$ a general null vector in a preferred frame (with $a \neq 0$), then the orthogonality condition with w gives

   $$v.w = a(w^1 - w^0) = 0 \quad \Rightarrow \quad w^0 = w^1 \ , \qquad (6)$$

   with w^2 and w^3 unconstrained. Therefore the square of w gives :

   $$\begin{aligned} w.w &= -(w^0)^2 + (w^1)^2 + (w^2)^2 + (w^3)^2 \\ &= (w^2)^2 + (w^3)^2 \geq 0 \quad \Leftrightarrow \quad \text{Not timelike} \end{aligned} \qquad (7)$$

   and is equal to zero if and only if $w^2 = w^3 = 0$ which implies $v = \lambda w$ for some constant λ.

(c) The sum of two timelike vectors is *not* necessarily timelike. As a counterexample, consider the vectors $u = (a, b, 0, 0)$ and $v = (-a, b, 0, 0)$; these are timelike for $|b| < |a|$ but their sum $u + v = (0, 2b, 0, 0)$ is clearly spacelike. What *is* true is that the sum of two future-pointing ($u^0 > 0, v^0 > 0$) timelike vectors is future-pointing and timelike.

Likewise, the sum of two spacelike vectors is *not* necessarily spacelike. As a counterexample, consider the vectors $u = (a, b, 0, 0)$ and $v = (a, -b, 0, 0)$; these are spacelike for $|b| > |a|$ but their sum $u + v = (2a, 0, 0, 0)$ is clearly timelike. This shows that one should not think that for spacelike vectors Minkowski geometry reduces to Euclidean geometry.

3. Tensor-Algebra: Lorentz-Tensoren

By definition a Lorentz vector transforms as

$$\bar{v}^\alpha = L^\alpha_\beta v^\beta \quad , \tag{8}$$

and a Lorentz covector as

$$\bar{u}_\alpha = \Lambda_\alpha^\beta u_\beta \quad , \tag{9}$$

with

$$\Lambda = (L^T)^{-1} \quad \Leftrightarrow \quad \Lambda_\alpha^\beta L^\alpha_\gamma = \delta^\beta_\gamma \quad . \tag{10}$$

This definition is such that the *contraction* between a vector and a covector is a *scalar* (invariant under Lorentz transformations),

$$\bar{u}_\alpha \bar{v}^\alpha = \Lambda_\alpha^\beta L^\alpha_\gamma u_\beta v^\gamma = \delta^\beta_\gamma u_\beta v^\gamma = u_\beta v^\beta = u_\alpha v^\alpha \quad . \tag{11}$$

Higher rank tensors transform like products of vectors and covectors, i.e. a (p, q) tensor transforms with p factors of L and q factors of Λ and is written as an object with p upper indices and q lower indices.

By the same calculation as above one then finds that any contracted pair of indices on a tensor is invariant (and therefore the tensor type of the resulting object can be read off just by looking at the number of uncontracted upper and lower indices). For example:

(a) for the contraction of a $(2, 0)$-tensor and a $(0, 1)$-tensor (covector) one has

$$\bar{T}^{\alpha\beta}\bar{u}_\beta = L^\alpha_\gamma L^\beta_\delta \, T^{\gamma\delta} \, \Lambda_\beta{}^\rho u_\rho = L^\alpha{}_\gamma \, \delta^\rho_\delta \, T^{\gamma\delta} u_\rho = L^\alpha{}_\gamma \left(T^{\gamma\delta} u_\delta \right) \tag{12}$$

so that $T^{\alpha\beta} u_\beta$ transforms like (and therefore is) a $(1, 0)$-tensor (vector).

(b) likewise the trace of a $(1, 1)$-tensor is a scalar,

$$\bar{T}^\alpha_\alpha = L^\alpha_\beta \Lambda_\alpha^\gamma T^\beta{}_\gamma = \delta^\gamma_\beta \, T^\beta{}_\gamma = T^\beta{}_\beta \tag{13}$$

Note that the trace of a $(0,2)$-tensor $T_{\alpha\beta}$ is not well-defined without using the Minkowski metric, i.e. something like

$$\mathsf{trace}(T_{\alpha\beta}) \stackrel{?}{=} \sum_\alpha T_{\alpha\alpha} \qquad (???) \tag{14}$$

is not Lorentz-invariant and therefore depends on the inertial system in which it is evaluated. However, with the help of the Minkowski metric one can define a Lorentz-invariant trace (i.e. a scalar) via

$$T_{\alpha\beta} \to T^{\alpha}_{\ \beta} = \eta^{\alpha\gamma} T_{\gamma\beta} \to T^{\alpha}_{\ \alpha} = \eta^{\alpha\gamma} T_{\gamma\alpha} \qquad \checkmark \tag{15}$$

("taking the trace with respect to η"). This is now manifestly a scalar.

4. Tensor-Analysis: Lorentz-Tensoren und ihre Ableitungen

The formalism is designed in such a way that the transformation behaviour (tensorial nature) can just be read off from the free indices. In particular, the partial derivative $(\partial/\partial x^\alpha) = \partial_\alpha$ transforms as a covector,

$$\bar{\partial}_\alpha \equiv \frac{\partial}{\partial \bar{x}^\alpha} = \Lambda_\alpha^{\ \beta} \partial_\beta \ \ . \tag{16}$$

Thus ∂_α acting on a (p,q)-tensor gives a $(p,q+1)$-tensor.

Then the answers are trivially (a) $\partial_\alpha f$ covector (b) $V^\alpha \partial_\alpha f$ scalar (c) $V^\alpha \partial_\beta f$ (1,1)-tensor (d) $\partial_\alpha V^\alpha$ scalar (e) $f \partial_\alpha V^\alpha$ scalar (f) $\partial_\alpha V_\beta$ (0,2)-tensor (g) $\partial_\alpha \partial_\beta f$ (0,2)-tensor (h) $V^\alpha \partial_\beta V_\alpha$ covector.