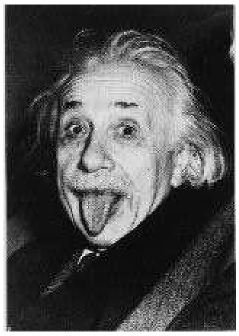

KFT Übungen 03 (Addendum)

Remarks on symmetrisation and anti-symmetrisation of tensors:

- A covariant 2-tensor $T_{\mu\nu}$, say, is said to be symmetric if $T_{\mu\nu} = T_{\nu\mu}$ and anti-symmetric if $T_{\mu\nu} = -T_{\nu\mu}$. This is well-defined because it is a Lorentz-invariant notion: a tensor is symmetric in all inertial systems iff it is symmetric in one inertial system, etc.
- This definition can be extended to any or all pairs of covariant indices or pairs of contravariant indices. Thus e.g. a tensor $T^{\mu_1\cdots\mu_p}$ is called totally symmetric (or totally anti-symmetric) if it is symmetric (anti-symmetric) under the exchange of any pair of indices. On the other hand, it is not meaningful to talk of the symmetry of a (1,1)-tensor, say, as an equation like $T^{\mu}_{\ \nu} = T^{\nu}_{\ \mu}$ is meaningless.
- The number of independent components of a general (p,q)-tensor is 4^{p+q}. The number of independent components is reduced if the tensor has some symmetry properties. Thus
  - a symmetric (0,2)- or (2,0)-tensor has $4 \times 5/2 = 10$ independent components,
  - an anti-symmetric (0,2)- or (2,0)-tensor has $4 \times 3/2 = 6$ independent components,
  - a totally anti-symmetric $(0,3)$-tensor $T_{\nu_1\ldots\nu_3}$ has $4 \times 3 \times 2/(2 \times 3) = 4$ independent components,
  - and a totally anti-symmetric $(0,4)$-tensor $T_{\nu_1\ldots\nu_4}$ has only got one independent component, namely T_{0123} (all the others being determined by anti-symmetry).
- Given any $(0,2)$-tensor $T_{\mu\nu}$, one can decompose it into its symmetric and anti-symmetric parts as

$$T_{\mu\nu} = \tfrac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) + \tfrac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \equiv T_{(\mu\nu)} + T_{[\mu\nu]} \quad . \tag{1}$$

  The decomposition into symmetric and anti-symmetric parts is Lorentz invariant. In particular, when $T_{\mu\nu}$ is a tensor, also $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ are tensors, and thus (anti-)symmetrisation is yet another linear operation that one can perform on tensors.

The factor $\frac{1}{2}$ is chosen such that the symmetrisation of a symmetric tensor is the same as the original tensor,

$$T_{\mu\nu} = T_{\nu\mu} \quad \Rightarrow \quad T_{(\mu\nu)} = T_{\mu\nu} \quad , \quad T_{[\mu\nu]} = 0 \tag{2}$$

(and likewise for the anti-symmetrisation of anti-symmetric tensors).

- This can be generalised to the (anti-)symmetrisation of any pair of (contravariant or covariant) indices; e.g.

$$T_{(\mu\nu)\lambda} = \tfrac{1}{2}(T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) \tag{3}$$

is the symmetrisation of $T_{\mu\nu\lambda}$ in its first and second index. It can also be generalised to the total (anti-)symmetrisation of a higher-rank tensor; e.g.

$$T_{(\mu\nu\lambda)} \equiv \tfrac{1}{3!}(T_{\mu\nu\lambda} + T_{\nu\mu\lambda} + T_{\lambda\nu\mu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu} + T_{\lambda\mu\nu}) \tag{4}$$

is totally symmetric, i.e. symmetric under the exchange of any pair of indices, and

$$T_{[\mu\nu\lambda]} \equiv \tfrac{1}{3!}(T_{\mu\nu\lambda} - T_{\nu\mu\lambda} - T_{\lambda\nu\mu} + T_{\nu\lambda\mu} - T_{\mu\lambda\nu} + T_{\lambda\mu\nu}) \tag{5}$$

is totally anti-symmetric. The prefactor $\frac{1}{6}$ is again there to ensure that the total symmetrisation of a totally symmetric tensor is the original tensor (and likewise for the total anti-symmetrisation of totally anti-symmetric tensors). This generalises in an evident way to higher rank p tensors, with the combinatorial prefactor $1/p!$.

- A special case, and the one of interest to us here, is the total anti-symmetrisation $T_{[\mu\nu\lambda]}$ of a tensor $T_{\mu\nu\lambda}$ that is already anti-symmetric in two of its indices, say $T_{\mu\lambda\nu} = -T_{\mu\nu\lambda}$. In that case, the 1st and 2nd terms of (5) are equal, as are the 3rd and 4th, and the 5th and 6th, and the formula (5) reduces to a sum of 3 terms,

$$T_{[\mu\nu\lambda]} = \tfrac{1}{3}(T_{\mu\nu\lambda} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu}) \ , \tag{6}$$

the sum of cyclic permutations of the 3 indices.

- In particular, the totally anti-symmetrised derivative of the Maxwell field strength tensor is

$$\partial_{[\alpha}F_{\beta\gamma]} = \tfrac{1}{3}(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}) \tag{7}$$

and therefore the homogeneous Maxwell equations can be written as

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad \Leftrightarrow \quad \partial_{[\alpha}F_{\beta\gamma]} = 0 \ . \tag{8}$$

From the above counting of components we learn (or reconfirm) that this equation has precisely 4 independent components, equal to the number of components of the homogeneous Maxwell equations.

- Since $\partial_{[\alpha}F_{\gamma\delta]}$ is totally anti-symmetric, nothing is lost by multiplying it by the totally anti-symmetric Levi-Civita symbol $\in^{\alpha\beta\gamma\delta}$ characterised (with a suitable choice of sign convention) by

$$\in^{\alpha\beta\gamma\delta}=\in^{[\alpha\beta\gamma\delta]} \quad , \quad \in^{0123}= -1 \ . \tag{9}$$

Thus the homogeneous Maxwell equations can equivalently be written as

$$\partial_{[\alpha}F_{\gamma\delta]} = 0 \quad \Leftrightarrow \quad \in^{\alpha\beta\gamma\delta} \partial_\alpha F_{\gamma\delta} = 0 \quad \Leftrightarrow \quad \partial_\alpha \tilde{F}^{\alpha\beta} = 0 \tag{10}$$

where

$$\tilde{F}^{\alpha\beta} = \tfrac{1}{2} \in^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{11}$$

is the *dual field strength tensor*.

- Essentially, $\tilde{F}^{\alpha\beta}$ is obtained from $F^{\alpha\beta}$ by the replacement $\vec{B} \to \vec{E}/c$ and $\vec{E}/c \to -\vec{B}$. Since under this replacement the left-hand sides of the inhomogeneous Maxwell equations get mapped to left-hand sides of the homogeneous Maxwell equations (electric-magnetic duality of the Maxwell equations), it is not surprising that the full set of (inhomogeneous and homogeneous) Maxwell equations can be written in the more symmetric and compact form

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad , \quad \partial_\alpha \tilde{F}^{\alpha\beta} = 0 \ . \tag{12}$$

These two sets of equations encapsulate all of electrodynamics (Maxwell theory).