Solutions to Assignments 08

1. Explicit Expression for the Riemann Curvature Tensor

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\lambda &= \nabla_\mu \left[\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\rho} V^\rho\right] - (\ \mu \leftrightarrow \nu\ ) \\
&= \partial_\mu \partial_\nu V^\lambda + \partial_\mu(\Gamma^\lambda_{\nu\rho} V^\rho) + \Gamma^\lambda_{\mu\rho}\partial_\nu V^\rho - \Gamma^\rho_{\mu\nu}\partial_\rho V^\lambda + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} V^\rho - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\
&= \partial_\mu(\Gamma^\lambda_{\nu\rho} V^\rho) + \Gamma^\lambda_{\mu\rho}\partial_\nu V^\rho + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\
&= (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \Gamma^\lambda_{\nu\rho}\partial_\mu V^\rho + \Gamma^\lambda_{\mu\rho}\partial_\nu V^\rho + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\
&= (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\
&= \left[\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right] V^\rho = R^\lambda{}_{\rho\mu\nu} V^\rho
\end{aligned} \tag{1}
$$

where in the third equality we dropped all the (μ, ν)-symmetric terms killed by the subtraction with the indices exchanged.

2. Properties of the Riemann Curvature Tensor

(a) We show that the fourth symmetry follows from (I),(II) and (III):

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= -(R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta}) = R_{\gamma\alpha\delta\beta} + R_{\delta\alpha\beta\gamma} \\
&= -(R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\alpha\delta}) - (R_{\delta\beta\gamma\alpha} + R_{\delta\gamma\alpha\beta}) \\
&= 2R_{\gamma\delta\alpha\beta} + R_{\beta\gamma\alpha\delta} + R_{\beta\delta\gamma\alpha} \\
&= 2R_{\gamma\delta\alpha\beta} - R_{\beta\alpha\delta\gamma} = 2R_{\gamma\delta\alpha\beta} - R_{\alpha\beta\gamma\delta} \\
&\Rightarrow R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}
\end{aligned} \tag{2}
$$

(b) From (a) we directly deduce the symmetry of the Ricci tensor :

$$
R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = R_{\rho\nu}{}^\rho{}_\mu = R^\rho{}_{\nu\rho\mu} = R_{\nu\mu} \tag{3}
$$

(c) Writing $\circlearrowleft$ for the cyclic permutations in (α, β, γ) and then using the third symmetry : $R^\rho{}_{\alpha\beta\gamma} + \circlearrowleft = 0$, we have :

$$
\begin{aligned}
[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft &= \nabla_\alpha(R^\lambda{}_{\rho\beta\gamma} V^\rho) - R^\lambda{}_{\rho\beta\gamma}\nabla_\alpha V^\rho + R^\rho{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\
&= \nabla_\alpha(R^\lambda{}_{\rho\beta\gamma}) V^\rho + R^\rho{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\
&= \nabla_\alpha(R^\lambda{}_{\rho\beta\gamma}) V^\rho + \circlearrowleft \\
&= g^{\lambda\mu}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] V^\nu = 0
\end{aligned} \tag{4}
$$

which gives the desired result.

(d) Contracting the Bianchi identity over the indices (μ, β) and (ν, α) one finds :

$$
\begin{aligned}
g^{\nu\alpha} g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] &= g^{\nu\alpha} g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}\right] \\
&= \nabla_\alpha R^{\alpha\beta}{}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}{}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}{}_{\alpha\beta} \\
&= -\nabla_\alpha R^\alpha{}_\gamma - \nabla_\beta R^\beta{}_\gamma + \nabla_\gamma R \\
&= -\nabla_\alpha\left[2R^\alpha{}_\gamma - \delta^\alpha_\gamma R\right] = 0
\end{aligned} \tag{5}
$$

And defining the *Einstein tensor* as $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$, we see that the contracted Bianchi identity (4) is equivalent to $\nabla^\alpha G_{\alpha\beta} = 0$ because :

$$\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R) = \frac{1}{2}\nabla_\alpha(2R^\alpha{}_\beta - g^\alpha{}_\beta R) \tag{6}$$

so that $\nabla^\alpha G_{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha\left[2R^\alpha{}_\gamma - \delta^\alpha_\gamma R\right] = 0$ where we have use the fact that $g^\alpha{}_\beta = g^{\alpha\lambda}g_{\lambda\beta} = \delta^\alpha{}_\beta$ simply because $g^{\alpha\lambda}$ is the inverse of $g_{\alpha\lambda}$.

3. The Geodesic Deviation Equation (section 8.3)

(a) This is obvious (I hope) since from expansion of the deplaced equation to first order one gets

$$\Gamma^\mu{}_{\nu\lambda}(x+\delta x)\tfrac{d}{d\tau}(x^\nu + \delta x^\nu)\tfrac{d}{d\tau}(x^\lambda + \delta x^\lambda) \\ = \partial_\rho\Gamma^\mu{}_{\nu\lambda}(x)\delta x^\rho \tfrac{d}{d\tau}x^\nu \tfrac{d}{d\tau}x^\lambda + 2\Gamma^\mu{}_{\nu\lambda}(x)\tfrac{d}{d\tau}x^\nu \tfrac{d}{d\tau}\delta x^\lambda \tag{7}$$

(the symmetry of the Christoffel symbols accounting for the factor of 2).

(b) Starting from

$$\frac{D}{D\tau}\delta x^\mu = \frac{d}{d\tau}\delta x^\mu + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\delta x^\lambda \tag{8}$$

one can calculate the 2nd derivative $D^2\delta x^\mu/D\tau^2$,

$$\begin{aligned}\frac{D^2}{D\tau^2}\dot{x}^\mu &= \frac{d}{d\tau}\left[\delta\dot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\delta x^\rho\right] + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\left[\delta\dot{x}^\beta + \Gamma^\beta{}_{\nu\rho}\dot{x}^\nu\delta x^\rho\right] \\ &= \delta\ddot{x}^\mu + (\partial_\lambda\Gamma^\mu{}_{\nu\rho})\dot{x}^\nu\dot{x}^\lambda\delta x^\rho + \Gamma^\mu{}_{\nu\rho}\ddot{x}^\nu\delta x^\rho + 2\Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\delta\dot{x}^\rho \\ &+ \Gamma^\mu{}_{\lambda\beta}\Gamma^\beta{}_{\nu\rho}\dot{x}^\lambda\dot{x}^\nu\delta x^\rho\end{aligned} \tag{9}$$

Subtracting from this the term $R^\mu_{\nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho$ and using the geodesic equation to eliminate $\ddot{x}^\nu$, one finds the equation

$$\tfrac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu{}_{\nu\lambda}(x)\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}\delta x^\lambda + \partial_\rho\Gamma^\mu{}_{\nu\lambda}(x)\delta x^\rho\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}x^\lambda = 0 \tag{10}$$