SOLUTIONS TO ASSIGNMENTS 07

1. ON THE MAXWELL EQUATIONS IN CURVED SPACE-TIME

The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \tag{1}$$

and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_{\beta}^{\ \gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \tag{2}$$

(a) The variation of the action with respect to the gauge field is

$$\begin{aligned} \delta S &= -\int \sqrt{g} d^4x \; (\partial_\mu \delta A_\nu) F^{\mu\nu} \\ &= -\int d^4x \; \partial_\mu (\sqrt{g} \delta A_\nu F^{\mu\nu}) + \int d^4x \; \partial_\mu (\sqrt{g} F^{\mu\nu}) \delta A_\nu \end{aligned} \tag{3}$$

where in the first line we have used the fact that there are 4 identical contributions to the variation. Then we note that the first term in the second line is a boundary term and vanishes because the variation vanishes on the boundary. Now using $\sqrt{g} \nabla_\mu F^{\mu\nu} = \partial_\mu (\sqrt{g} F^{\mu\nu})$ we are left with

$$\delta S = \int \sqrt{g} d^4x \; (\nabla_\mu F^{\mu\nu}) \delta A_\nu \tag{4}$$

which gives the equations of motion.

(b) We compute :

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu (F^{\mu}_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) \\ &= (\nabla_\mu F^{\mu}_{\ \lambda}) F^{\nu\lambda} + F^{\mu}_{\ \lambda} \nabla_\mu F^{\nu\lambda} - \tfrac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\ &= -J_\lambda F^{\nu\lambda} + F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2} \nabla^\nu F^{\mu\lambda} \right) \\ &= J_\lambda F^{\lambda\nu} + \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right) \end{aligned} \tag{5}$$

Then we rewrite the term $\frac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda}$ in a different way by relabeling the indices and using the anti-symmetry of $F_{\mu\nu}$:

$$\tfrac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda} = \tfrac{1}{2} F_{\lambda\mu} \nabla^\lambda F^{\nu\mu} = -\tfrac{1}{2} F_{\mu\lambda} \nabla^\lambda F^{\nu\mu} \tag{6}$$

so that we can now use $\nabla_{[\lambda} F_{\mu\nu]} = 0$ to have at the end :

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= J_\lambda F^{\lambda\nu} - \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) \\ &= J_\lambda F^{\lambda\nu} \end{aligned} \tag{7}$$

(c) For the metric variation of the action, we can also use the general formula

$$\delta S = \int d^4x\,(\delta(\sqrt{g})L + \sqrt{g}\delta L) = -\tfrac{1}{2}\int d^4x\sqrt{g}\,(g_{\mu\nu}L\delta g^{\mu\nu} - 2\delta L) \qquad (8)$$

(valid for any Lagrangian L). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}) = 2(\delta g^{\mu\lambda})g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = 2(\delta g^{\mu\nu})g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} = 2(\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda} \qquad (9)$$

and therefore $-2\delta L = (\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda}$. Putting the pieces together one gets (2).

2. KRUSKAL COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME: SOLUTION I (DIRECT CALCULATION USING THE COORDINATE TRANSFORMATION)

To compute the Schwarzschild metric in the new (X,T)-coordinate, it's useful to consider the two expression $t(X,T)$ and $r^*(X,T)$. To find these we first rewrite (X,T) as :

$$\begin{aligned} X &= \mathrm{e}^{r^*/4m}\cosh(t/4m) \\ T &= \mathrm{e}^{r^*/4m}\sinh(t/4m)\ . \end{aligned} \qquad (10)$$

This leads in particular to :

$$X^2 - T^2 = \mathrm{e}^{r^*/2m} = \mathrm{e}^{r/2m}\left(\frac{r}{2m} - 1\right) = rF(r)\frac{\mathrm{e}^{r/2m}}{2m} \qquad (11)$$

which is a way to express r implicitly (we have defined $F := \frac{\partial r}{\partial r*} = 1 - \frac{2m}{r}$ for later use). Now, from (6) it also follows that :

$$\begin{aligned} t &= 4m\,\mathrm{atanh}\,(T/X) \\ r^* &= 2m\log\left(X^2 - T^2\right)\ . \end{aligned} \qquad (12)$$

and this allows us to compute the partial derivative we will need :

$$\begin{aligned} \frac{\partial t}{\partial T} &= \frac{4mX}{X^2 - T^2} \qquad & \frac{\partial r}{\partial T} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial T} = F\frac{4mT}{T^2 - X^2} \\ \frac{\partial t}{\partial X} &= \frac{4mT}{T^2 - X^2} \qquad & \frac{\partial r}{\partial X} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial X} = F\frac{4mX}{X^2 - T^2} \end{aligned} \qquad (13)$$

Then it is straightforward to compute the Schwarzschild metric starting from the old (t,r)-coordinate and we get :

$$\begin{aligned} ds^2 &= -F dt^2 + F^{-1} dr^2 + r^2 d\Omega^2 \\ &= -F\left(\frac{\partial t}{\partial T}dT + \frac{\partial t}{\partial X}dX\right)^2 + F^{-1}\left(\frac{\partial r}{\partial T}dT + \frac{\partial r}{\partial X}dX\right)^2 + r^2 d\Omega^2 \\ &= \frac{16m^2 F}{(X^2-T^2)^2}\left[-\left(XdT - TdX\right)^2 + \left(-TdT + XdX\right)^2\right] + r^2 d\Omega^2 \\ &= \frac{16m^2 F}{(X^2-T^2)}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \\ &= \frac{32m^3}{r}\,\mathrm{e}^{-r/2m}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \end{aligned} \tag{14}$$

where in the last step we have used (8).

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution II (massaging the metric into a convenient form)

   Write the Schwarzschild metric as

$$ds^2 = (1-2m/r)[-dt^2 + dr^{*2}] + r^2 d\Omega^2 = (1-2m/r)[-du\, dv] + r(u,v)^2 d\Omega^2 \tag{15}$$

where $r^* = r + 2m\log(r/2m - 1)$ is the tortoise coordinate, and $v = t + r^*$, $u = t - r^*$ are the "advanced" and "retarded" Eddington-Finkelstein coordinates. Now note that

$$\frac{v-u}{4m} = \frac{r}{2m} + \log\left(\frac{r}{2m} - 1\right) \quad , \tag{16}$$

so that

$$1 - \frac{2m}{r} = \frac{2m}{r}\left(\frac{r}{2m} - 1\right) = \frac{2m}{r}\mathrm{e}^{-r/2m}\mathrm{e}^{(v-u)/4m} \quad . \tag{17}$$

Thus the metric is

$$ds^2 = \frac{2m}{r}\mathrm{e}^{-r/2m}\left(\mathrm{e}^{v/4m}dv\right)\left(-\mathrm{e}^{-u/4m}du\right) + r(u,v)^2 d\Omega^2 \quad . \tag{18}$$

Therefore it is natural to introduce

$$V = \mathrm{e}^{v/4m} \quad , \quad U = -\mathrm{e}^{-u/4m} \quad , \tag{19}$$

and T and X via $V = T + X, U = T - X$, so that

$$\begin{aligned} ds^2 &= -\frac{32m^3}{r}\mathrm{e}^{-r/2m} dU dV + r(u,v)^2 d\Omega^2 \\ &= \frac{32m^3}{r}\mathrm{e}^{-r/2m}[-dT^2 + dX^2] + r(T,X)^2 d\Omega^2 \quad . \end{aligned} \tag{20}$$

Moreover, equation (17) implies that

$$\left(\frac{r}{2m} - 1\right)\mathrm{e}^{r/2m} = \mathrm{e}^{(v-u)/4m} = -UV = X^2 - T^2 \quad , \tag{21}$$

so that $r = 2m \Leftrightarrow X = \pm T$ and $r = 0 \Leftrightarrow X^2 - T^2 = -1$.

3. Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

(a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates (t, r) defining a new coordinate $T(t, r) = t + \psi(r)$. This leads us to rewrite the metric with $dT = dt + \psi' dr$ and we find :

$$ds^2 = -f(r)dT^2 + 2f(r)\psi' dTdr + f(r)^{-1}(1 - f(r)^2\psi'^2)dr^2 + r^2 d\Omega^2 \tag{22}$$

Choosing $C(r) = f(r)\psi'$ gives the desired result and the function $C(r)$ is completely arbitrary because $\psi(r)$ is arbitrary.

(b) In the Painlevé-Gullstrand coordinate we make a particular choice for $C(r)$ namely $C(r) = \sqrt{1 - f(r)}$ such that $g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1$. We are thus left with the metric :

$$ds^2 = -f(r)dT^2 + 2\sqrt{\frac{2m}{r}}dTdr + dr^2 + r^2 d\Omega^2 \tag{23}$$

Now, with this new choice of coordinate we sees that any component $g_{\mu\nu}$ of the metric stays finite for any value of $r > 0$ (it was not the case at the beginning in the (t, r)-coordinate). In addition to that we also notice that the determinant of the metric is :

$$\det(g_{\mu\nu}) = (-f(r) - \frac{2m}{r})r^4 \sin(\theta)^2 = -r^4 \sin(\theta)^2 \tag{24}$$

which is non-vanishing for any $r > 0$ (with $\theta \neq \pm 0, \pi$ of course).

(c) If we now make the choice $C(r) = 1$, then the metric becomes :

$$ds^2 = -f(r)dT^2 + 2dTdr + r^2 d\Omega^2 \tag{25}$$

and if we rename $T(t, r) \rightarrow v(t, r)$, then :

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{26}$$

which is exactly the metric in the Eddington-Finkelstein coordinates. We can check explicitly that the transformation is indeed the same. The particular choice $C(r) = 1$ implies that $\psi(r)$ is such that $\psi' = \frac{1}{f(r)}$. Thus $\psi = r^* + c$ where the constant can be set to zero so that we have $T(t, r) = t + \psi = t + r^* = v(t, r)$.