Solutions to Assignments 06

1. Stationary and Freely Falling Schwarzschild Observers

   (a) The observer is sitting at fixed radius and angles, therefore his worldline 4-velocity is of the form :

$$\frac{dx^\mu}{d\tau} = u^\mu = \begin{pmatrix} u^t \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ with: } u_\mu u^\mu = -1, \text{ and } u^t > 0 \Rightarrow u^t = \frac{1}{\sqrt{1-\frac{2m}{r}}} \quad (1)$$

   The acceleration is then

$$\begin{aligned} a^\mu = \nabla_\tau u^\mu &= u^\rho \nabla_\rho u^\mu \\ &= u^t \partial_t u^\mu + u^t \Gamma^\mu_{tt} u^t \\ &= \Gamma^\mu_{tt} \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} g^{\mu\rho} \partial_\rho g_{tt} \frac{1}{1-\frac{2m}{r}} \\ \text{for } \mu \neq r &= 0 \\ \text{for } \mu = r &= \frac{1}{2} g^{rr} \partial_r (1 - \frac{2m}{r}) \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} \partial_r \frac{2m}{r} = \frac{m}{r^2} \end{aligned} \quad (2)$$

   and therefore the norm of the acceleration is

$$\begin{aligned} g_{\mu\nu} a^\mu a^\nu &= g_{rr} a^r a^r \\ &= \frac{1}{1-\frac{2m}{r}} \frac{m^2}{r^4} \quad . \end{aligned} \quad (3)$$

   Note that this approaches the Newtonian value $(m/r^2)^2$ for $r \to \infty$, while the required acceleration to keep the stationary observer at rest diverges as $r \to 2m$.

   (b) Outline of the derivation (see sections 13.1 and 13.8 of the lecture notes for details). For zero angular momentum, and with $\dot{r}_{r=R} = 0$ the effective potential equation reduces to

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} \quad \Rightarrow \quad \dot{r}^2 = \frac{2m}{r} - \frac{2m}{R} \quad , \quad (4)$$

   which integrates to

$$\tau_{R \to r_1} = -(2m)^{-1/2} \int_R^{r_1} dr \left(\frac{Rr}{R-r} \right)^{1/2} \quad . \quad (5)$$

This integral can be calculated in closed form, e.g. via the change of variables

$$\frac{r}{R} = \sin^2 \alpha \qquad \alpha_1 \leq \alpha \leq \frac{\pi}{2} \quad , \tag{6}$$

leading to

$$\tau_{R\to r_1} = 2\left(\frac{R^3}{2m}\right)^{1/2} \int_{\alpha_1}^{\pi/2} d\alpha\ \sin^2 \alpha = \left(\frac{R^3}{2m}\right)^{1/2} \left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{\alpha_1}^{\pi/2} \quad . \tag{7}$$

For $r_1 \to 0 \Leftrightarrow \alpha_1 \to 0$ one obtains

$$\tau_{R\to 0} = \left(\frac{R^3}{2m}\right)^{1/2} (\pi/2) = \pi \left(\frac{R^3}{8m}\right)^{1/2} \tag{8}$$

R and $r_S = 2m$ have dimensions of length, thus the quantity above also has dimensions of length, so what we have actually calculated is $c\tau$, not τ. To obtain proper time, we thus need to divide by c. Using the approximate values

$$(R)_{\text{sun}} \approx 7 \times 10^{10}\mathsf{cm} \quad (2\mathsf{m})_{\text{sun}} \approx 3 \times 10^{5}\mathsf{cm} \quad \mathsf{c} \approx 3 \times 10^{10}\mathsf{cm\ s}^{-1} \tag{9}$$

one finds $\tau_{R\to 0} \approx 2 \times 10^3\mathsf{s}$, which is roughly 30 minutes.

2. Rindler Coordinates and the Schwarzschild geometry near $r = r_S$

The approximate geometry (radial part only) is given by the metric

$$ds^2 = -\frac{r-2m}{2m}dt^2 + \frac{2m}{r-2m}dr^2 \quad . \tag{10}$$

Thus the proper radial length ρ is given by

$$d\rho = \sqrt{\frac{2m}{r-2m}}dr \tag{11}$$

which can be integrated to $\rho = 2\sqrt{2m}\sqrt{r-2m}$ such that $\rho^2 = 8m(r-2m)$. Moreover, using $\eta = \frac{t}{4m}$, thus $d\eta = \frac{1}{4m}dt$, one finds

$$\begin{aligned} ds^2 &= -\frac{r-2m}{2m}dt^2 + \frac{2m}{r-2m}dr^2 \\ &= -16m^2\frac{r-2m}{2m}d\eta^2 + d\rho^2 \\ &= -\rho^2 d\eta^2 + d\rho^2 \end{aligned} \tag{12}$$

for the near-r_S Schwarzschild metric. This is just the Rindler metric. The stationary observers of exercise 1(a) correspond to the constantly accelerating observers at $\rho = \rho_0$ constant, while the freely falling observers of exercise 1(b) correspond to inertial observers in Minkowski space.

3. On the Klein-Gordon Field in a curved space-time

The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \tag{13}$$

and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} L \tag{14}$$

(a) Variation of the mass term evidently gives rise to $-\int \sqrt{g} d^4x m^2 \phi \delta\phi$, so we focus on the kinetic term. Thinking of the partial derivatives in the action as covariant derivatives, and using $\nabla_\alpha g_{\mu\nu} = 0$, $\nabla_\alpha \sqrt{g} = 0$, one can integrate by parts to one's heart's content as in Minkowski space to find

$$\delta S = -\int \sqrt{g} d^4x \; g^{\mu\nu} \nabla_\mu \delta\phi \nabla_\nu \phi = \int \sqrt{g} d^4x \; g^{\mu\nu} (\nabla_\mu \nabla_\nu \phi) \delta\phi \tag{15}$$

Combining the variations of the kinetic and mass terms, one deduces the equation of motion $(\Box - m^2)\phi = 0$. If one works with the partial derivatives, one of course finds the same equation, but this time with the Laplace operator in the form $\sqrt{g}\Box = \partial_\alpha(\sqrt{g} g^{\alpha\beta} \partial_\beta)$.

(b) To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu \phi = \nabla_\mu \phi$, the commutativity $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$ of the covariant derivative on scalars, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu \nabla_\mu \phi = m^2 \phi$, then the result follows:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) + \nabla^\mu (g_{\mu\nu} L) \\ &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) - \frac{1}{2} \nabla_\nu (\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2) \\ &= \partial_\nu \phi \nabla^\mu \partial_\mu \phi + \partial_\mu \phi \nabla^\mu \partial_\nu \phi - \partial_\lambda \phi \nabla_\nu \partial^\lambda \phi - m^2 \phi \nabla_\nu \phi \\ &= \partial_\nu \phi \; m^2 \phi + \partial_\mu \phi \nabla^\mu \nabla_\nu \phi - \partial_\lambda \phi \nabla^\lambda \nabla_\nu \phi - m^2 \phi \nabla_\nu \phi \\ &= 0 \end{aligned} \tag{16}$$

(c) The variation of the action with respect to the metric is

$$\delta S = \int d^4x \, (\delta(\sqrt{g}) L + \sqrt{g} \delta L) = -\tfrac{1}{2} \int d^4x \sqrt{g} \, (g_{\mu\nu} L \delta g^{\mu\nu} - 2\delta L) \tag{17}$$

(valid for any Lagrangian L). Using $\delta(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = (\delta g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi$, one finds

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} L + \partial_\mu \phi \partial_\nu \phi) \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} \tag{18}$$

as claimed.