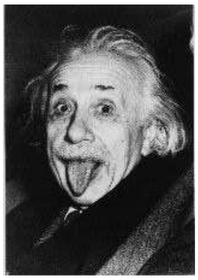

GR Assignments 08

Tensor Analysis IV: the Riemann Curvature Tensor

1. Explicit Expression for the Riemann Curvature Tensor

   One of the characteristic properties of the covariant derivative ∇_μ is that *second covariant derivatives acting on scalars commute*, $[\nabla_\mu, \nabla_\nu]\phi = 0$ (as we have seen, this follows from the symmetry of the Christoffel symbols). This, however, is no longer true when covariant derivatives act on tensors of higher rank. In general, *second covariant derivatives acting on vectors do not commute.* However, the commutator $[\nabla_\mu, \nabla_\nu]V^\lambda$ turns out to depend only linearly on V and not on its derivatives, i.e. one has

   $$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\sigma\mu\nu}V^\sigma \tag{1}$$

   for some collection of objects $R^\lambda{}_{\sigma\mu\nu}$. Since everything on the left and V on the right hand side of this equation are tensors, also the $R^\lambda{}_{\sigma\mu\nu}$ are the components of a tensor, namely the famous *Riemann(-Christoffel) Curvature Tensor*. By explicit calculation show that the Riemann tensor is given by

   $$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\lambda_{\sigma\nu} - \partial_\nu\Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma} \quad . \tag{2}$$

2. Properties of the Riemann Curvature Tensor

   In the course, we defined the contractions of the Riemann curvature tensor, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries

   $$\begin{aligned} &(I) \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \qquad (II) \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \\ &(III) \quad R_{\alpha[\beta\gamma\delta]} = 0 \;\Leftrightarrow\; R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \end{aligned} \tag{3}$$

   [see section 7.3 of the lecture notes for proofs and make sure that you understand the details!]

   (a) Show that the above symmetries imply the property

   $$(IV) \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \quad . \tag{4}$$

   (b) Show that symmetry (IV) implies that the Ricci tensor is symmetric.

(c) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0 \ . \tag{5}$$

[If you like, you can check this explicitly by writing out all the commutators.] Show that this, together with the definition (1), implies the *Bianchi identity*

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in } (\alpha, \beta, \gamma) = 0 \tag{6}$$

(d) By contracting the Bianchi identity over the indices (μ, β) (multiplication by $g^{\mu\beta}$) and (ν, α) (multiplication by $g^{\nu\alpha}$), deduce the identity (*contracted Bianchi identity*)

$$\nabla_\alpha (2R^\alpha_{\ \gamma} - \delta^\alpha_{\ \gamma} R) = 0 \tag{7}$$

and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$.

Remark: In all these equations indices are lowered and raised with the metric and its inverse: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda} R^\lambda_{\ \beta\gamma\delta}$, $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$, $R^\alpha_{\ \gamma} = g^{\alpha\beta} R_{\beta\gamma}$ etc.

3. The Geodesic Deviation Equation (section 8.3)

(a) Consider two geodesics $x^\mu(\tau)$ and $x^\mu(\tau) + \delta x^\mu(\tau)$,

$$\begin{aligned} &\frac{D}{D\tau}\dot{x}^\mu \equiv \tfrac{d^2}{d\tau^2} x^\mu + \Gamma^\mu_{\ \nu\lambda}(x) \tfrac{d}{d\tau} x^\nu \tfrac{d}{d\tau} x^\lambda = 0 \ , \\ &\tfrac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu_{\ \nu\lambda}(x + \delta x) \tfrac{d}{d\tau}(x^\nu + \delta x^\nu) \tfrac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0 \ . \end{aligned} \tag{8}$$

with $\delta x^\mu(\tau)$ an infinitesimal *deviation vector*. Show that this implies the evolution equation

$$\tfrac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu_{\ \nu\lambda}(x) \tfrac{d}{d\tau} x^\nu \tfrac{d}{d\tau} \delta x^\lambda + \partial_\rho \Gamma^\mu_{\ \nu\lambda}(x) \delta x^\rho \tfrac{d}{d\tau} x^\nu \tfrac{d}{d\tau} x^\lambda = 0 \tag{9}$$

for the deviation vector $\delta x^\mu(\tau)$.

(b) Show that (9) can be written in manifestly covariant form as

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\ \nu\lambda\rho} \dot{x}^\nu \dot{x}^\lambda \delta x^\rho \ . \tag{10}$$

Hint: It is easier to start from (10) and to deduce (9). Also: you will of course have to use the fact that $x^\mu(\tau)$ itself satisfies the geodesic equation.