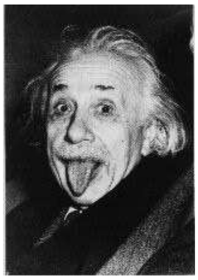

GR Assignments 07

1. ON THE MAXWELL EQUATIONS IN CURVED SPACE-TIME

   The Maxwell action in a gravitational background is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta}F^{\alpha\beta} \tag{1}$$

   The gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \tfrac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \tag{2}$$

   (a) Derive the vacuum Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0 \tag{3}$$

   by variation of the action with respect to the gauge field A_μ.

   Remark: You can write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ either in terms of partial or in terms of covariant derivatives. In the former case, you will need to use the equation $\nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu})$ for the covariant divergence in terms of $\sqrt{g}$ etc. which you derived before. In the latter case, you can use the fact that $\sqrt{g}$ is covariantly constant ($\nabla_\mu\sqrt{g} = 0$).

   (b) Use the Maxwell equations (with a source),

$$\nabla_\mu F^{\mu\nu} = -J^\nu \;\; , \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0 \;\Leftrightarrow\; \nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0 \tag{4}$$

   to deduce the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \;\; . \tag{5}$$

   Remark: This is a tensorial equation. For such calculations you should just use the properties of the covariant derivative and **not** write out the covariant derivative in terms of the non-tensorial Christoffel symbols and partial derivatives.

   Hint: Instead of embarking blindly on this calculation, remind yourself first how to do the calculation in Minkowski space. Exactly the same procedure should then work in general. If done correctly, this should be a four-line calculation.

(c) Show that the energy momentum tensor is related to the variation of the Maxwell action with respect to the metric in the same way as for the scalar field.

Hint: don't forget the implicit metric-dependence in expressions like $F_{\alpha\beta}F^{\alpha\beta}$.

2. KRUSKAL COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME

To get to a completely non-singular form of the Schwarzschild metric (for any $r \neq 0$, not just near $r = r_S$), introduce the so-called *Kruskal coordinates*

$$\begin{aligned} X &= \tfrac{1}{2}(\mathrm{e}^{(t+r^*)/4m} + \mathrm{e}^{-(t-r^*)/4m}) \\ T &= \tfrac{1}{2}(\mathrm{e}^{(t+r^*)/4m} - \mathrm{e}^{-(t-r^*)/4m}) \ . \end{aligned} \tag{6}$$

where the *tortoise coordinate* r^* is defined by $r^* = r + 2m\log(r/2m - 1)$.

Show that in terms of these coordinates the metric is

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}(-dT^2 + dX^2) + r^2 d\Omega^2 \ , \tag{7}$$

where $r = r(T, X)$ is implicitly given by

$$X^2 - T^2 = (r/2m - 1)\mathrm{e}^{r/2m} \ . \tag{8}$$

3. PAINLEVÉ-GULLSTRAND COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME

In the Schwarzschild coordinates (t, r), the Schwarzschild metric has the standard form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \qquad\qquad f(r) = 1 - \frac{2m}{r} \ . \tag{9}$$

(a) Show that the metric

$$ds^2 = -f(r)dT^2 + 2C(r)dTdr + f(r)^{-1}(1 - C(r)^2)dr^2 + r^2 d\Omega^2 \tag{10}$$

is equivalent to the Schwarzschild metric for *any* function $C(r)$. [**Hint:** Begin with (9) and consider the coordinate transformation $T(t, r) = t + \psi(r)$.]

(b) Now choose $C(r)$ such that $g_{rr} = 1$ (Painlevé-Gullstrand (PG) coordinates). Write down the resulting metric and show that it is completely non-singular for all $r > 0$ (in particular for $r \to 2m$), i.e. show that the metric coefficients are bounded and the determinant is non-zero.

(c) Show that the choice $C(r) = 1$ gives rise to the metric in Eddington-Finkelstein coordinates (with $T \equiv v = t + r^*$).

Optional Further Exercises:

Test your understanding/knowledge of GR (solutions will *not* be provided).

The metric in PG coordinates is related to timelike geodesics in the same way as the metric in Eddington-Finkelstein coordinates is related to null geodesics. To see this, consider the field of normal vectors $u_\alpha = -\partial_\alpha T$ orthogonal to the surfaces of constant T (in Scharzschild coordinates $x^\alpha = (t, r, \ldots)$).

(d) Show that $u^\alpha u_\alpha = -1$. Then show that in general the two properties $u^\alpha u_\alpha =$ const and $u_\alpha = -\partial_\alpha T$ imply that u^α is geodesic, i.e. $u^\beta \nabla_\beta u^\alpha = 0$.

(e) Show that the geodesics $x^\alpha(\tau)$ to which the u^α are tangent ($u^\alpha = \dot{x}^\alpha$) are radial geodesics ($L = 0$) with proper time $\tau = T$ and energy $E = 1$ (corresponding to observers that would have started off at rest at $r = \infty$).