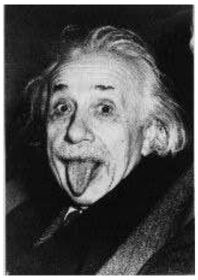

GR ASSIGNMENTS 06

1. STATIONARY AND FREELY FALLING SCHWARZSCHILD OBSERVERS

   (a) Consider a stationary observer (sitting at fixed values of $(r > 2m, \theta, \phi)$) in the Schwarzschild geometry

   $$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \qquad (1)$$

   Determine his worldline 4-velocity $u^\alpha = dx^\alpha/d\tau$ and the acceleration $a^\alpha = \nabla_\tau u^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ and calculate $g_{\alpha\beta}a^\alpha a^\beta$. What happens as $r \to \infty$ and $r \to 2m$?

   (b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of c) given by

   $$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} \quad . \qquad (2)$$

   Estimate this for R the radius of the sun ($R \sim 7 \times 10^{10}$ cm) and $2m$ its Schwarzschild radius ($2m \sim 3 \times 10^5$ cm), restoring the correct factors of c, and show that this is of the order of an hour.

   Remark: this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.

2. RINDLER COORDINATES AND THE SCHWARZSCHILD GEOMETRY NEAR $r = r_S$

   Recall that in Rindler coordinates (ρ, η),

   $$x^0 = \rho \sinh\eta \qquad x^1 = \rho\cosh\eta \quad . \qquad (3)$$

   the 2-dimensional Minkowski metric $ds^2 = -(dx^0)^2 + (dx^1)^2$ takes the form

   $$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \quad , \qquad (4)$$

   and the curves $\rho = \rho_0$ constant are the worldlines of Minkowski observers with constant acceleration $\mathsf{a} = 1/\rho_0$.

The purpose of this exercise is to show that the (t,r)-part of the geometry of the Schwarzschild metric near the Schwarzschild radius $r = r_S \equiv 2m$ has exactly the above form (with $r \to r_S \Leftrightarrow \rho \to 0$) establishing that the geometry is non-singular at $r = r_S$ and providing insight into the physics of the Schwarzschild metric and coordinates.

To that end consider the near-r_S geometry of the Schwarzschild metric, defined by approximating $(1-2m/r)$ and its inverse by

$$1-\frac{2m}{r} = \frac{r-2m}{r} \approx \frac{r-2m}{2m} \quad , \quad \left(1-\frac{2m}{r}\right)^{-1} \approx \frac{2m}{r-2m} \quad . \tag{5}$$

Introduce a new radial coordinate ρ as the proper radial distance from $r = r_S$ in the approximate geometry defined by the above equations, and the rescaled time-coordinate $\eta = t/4m$. Show that $\rho^2 = 8m(r-2m)$ and that the (t,r)-part of the near-r_S Schwarzschild metric takes precisely the above Rindler form (4), and interpret the stationary and freely falling Schwarzschild observers from the previous exercise from this Rindler point of view.

3. On the Klein-Gordon Field in a Curved Space-Time

   The action of a real (free, massice) scalar field ϕ in a gravitational background $g_{\alpha\beta}$ is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{6}$$

   The corresponding generally covariant energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{7}$$

   (a) Derive the equation of motion

$$\left(\Box - m^2\right)\phi = 0 \tag{8}$$

   ($\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$) for ϕ from variation of the action (6) with respect to ϕ.

   (b) Show that $T_{\alpha\beta}$ is conserved when ϕ is a solution to the Klein-Gordon equation of motion,

$$\left(\Box - m^2\right)\phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \; . \tag{9}$$

   (c) Show that $T_{\alpha\beta}$ is related to the variation of the action *with respect to the metric* by

$$\delta S = -\tfrac{1}{2} \int \sqrt{g} d^4x \; T_{\alpha\beta}\delta g^{\alpha\beta} \tag{10}$$

   Hint: use the variational formula from the exercises of week 05, as well as the (hopefully evident) identity $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$.