Solutions to Assignments 06

1. Scalar Fields in a Gravitational Field

   The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \tag{1}$$

   and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} L \tag{2}$$

   (a) To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu \phi = \nabla_\mu \phi$, the commutativity $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$ of the covariant derivative on scalars, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu \nabla_\mu \phi = m^2 \phi$, then the result follows:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) + \nabla^\mu (g_{\mu\nu} L) \\ &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) - \frac{1}{2} \nabla_\nu (\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2) \\ &= \partial_\nu \phi \nabla^\mu \partial_\mu \phi + \partial_\mu \phi \nabla^\mu \partial_\nu \phi - \partial_\lambda \phi \nabla_\nu \partial^\lambda \phi - m^2 \phi \nabla_\nu \phi \\ &= \partial_\nu \phi \; m^2 \phi + \partial_\mu \phi \nabla^\mu \nabla_\nu \phi - \partial_\lambda \phi \nabla^\lambda \nabla_\nu \phi - m^2 \phi \nabla_\nu \phi = 0 \; . \end{aligned} \tag{3}$$

   (b) The variation of the action with respect to the metric is

$$\delta S = \int d^4x \left(\delta(\sqrt{g}) L + \sqrt{g} \delta L \right) = -\tfrac{1}{2} \int d^4x \sqrt{g} \left(g_{\mu\nu} L \delta g^{\mu\nu} - 2 \delta L \right) \tag{4}$$

   (valid for any Lagrangian L). Using $\delta(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = (\delta g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi$, one finds

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} L + \partial_\mu \phi \partial_\nu \phi) \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} \tag{5}$$

   as claimed.

2. Maxwell Equations in a Gravitational Field

   The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \tag{6}$$

   and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta^{\;\gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \tag{7}$$

   (a) We compute, using $\nabla_\mu F^{\mu\lambda} = 0$,

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu (F^\mu_{\;\lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) = F^\mu_{\;\lambda} \nabla_\mu F^{\nu\lambda} - \tfrac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\ &= F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2} \nabla^\nu F^{\mu\lambda} \right) = \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right) \end{aligned} \tag{8}$$

   Uinsg the anti-symmetry of F, one can write this as

$$\nabla_\mu T^{\mu\nu} = -\tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) = 0 \tag{9}$$

(b) For the metric variation of the action, we can also use the general formula (4). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}) = 2(\delta g^{\mu\lambda})g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = 2(\delta g^{\mu\nu})g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} = 2(\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda} \tag{10}$$

and therefore $-2\delta L = (\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda}$. Thus (7) follows.

Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

(a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates (t, r) defining a new coordinate $T(t, r) = t + \psi(r)$. This leads us to rewrite the metric with $dT = dt + \psi' dr$ and we find

$$ds^2 = -f(r)dT^2 + 2f(r)\psi'(r)dTdr + f(r)^{-1}(1 - f(r)^2\psi'(r)^2)dr^2 + r^2d\Omega^2 \tag{11}$$

Choosing $C(r) = f(r)\psi'(r)$ gives the desired result and the function $C(r)$ is completely arbitrary because $\psi(r)$ is arbitrary.

(b) For the Painlevé-Gullstrand coordinate we make a particular choice for $C(r)$ namely $C(r) = \sqrt{1 - f(r)}$ such that $g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1$. We are thus left with the metric

$$ds^2 = -f(r)dT^2 + 2\sqrt{\frac{2m}{r}}dTdr + dr^2 + r^2d\Omega^2 \tag{12}$$

Now, with this new choice of coordinate we sees that any component $g_{\mu\nu}$ of the metric stays finite for any value of $r > 0$ (this was not the case at the beginning in the (t, r)-coordinates). In addition to that we also notice that the determinant of the metric is :

$$\det(g_{\mu\nu}) = (-f(r) - \frac{2m}{r})r^4\sin(\theta)^2 = -r^4\sin(\theta)^2 \tag{13}$$

which is non-vanishing for any $r > 0$ (with $\theta \neq 0, \pi$ of course).

(c) If we now make the choice $C(r) = 1$, then the metric becomes :

$$ds^2 = -f(r)dT^2 + 2dTdr + r^2d\Omega^2 \tag{14}$$

and if we rename $T(t, r)$ to $v(t, r)$, then

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2 \tag{15}$$

is exactly the metric in the Eddington-Finkelstein coordinates.

We can also check explicitly that the coordinate transformation is indeed also the same. The particular choice $C(r) = 1$ implies that $\psi(r)$ is such that

$$C(r) = 1 \quad \Leftrightarrow \quad \psi'(r) = \frac{1}{f(r)} \quad \Leftrightarrow \quad \psi(r) = r^* + c \ , \tag{16}$$

where the constant c can be set to zero so that we have

$$T(t, r) = t + \psi(r) = t + r^* = v(t, r) \tag{17}$$

as we should.