SOLUTIONS TO ASSIGNMENTS 05

1. TENSOR ANALYSIS II: THE COVARIANT DIVERGENCE AND THE LAPLACIAN

   (a) The covariant divergence is $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$ where

   $$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} \tag{1}$$

   (the 1st and 3rd term cancel). Now we use $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find

   $$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} = \frac{1}{2} g^{-1}\partial_\lambda g = g^{-1/2}\partial_\lambda g^{+1/2} \tag{2}$$

   where in the last equality we used the fact that $\partial_\lambda g^{+1/2} = \frac{1}{2} g^{-1/2}\partial_\lambda g$. We can now compute the covariant divergence

   $$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\lambda} J^\lambda = \partial_\mu J^\mu + J^\lambda g^{-1/2}\partial_\lambda g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} J^\mu) \ . \tag{3}$$

   (b) Analogously, for the covariant divergence of an anti-symmetric $(2,0)$-tensor $F^{\mu\nu}$ one has, using (3),

   $$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2}\partial_\rho g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} F^{\mu\nu}) \end{aligned} \tag{4}$$

   where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^\nu_{\mu\rho}$).
   Finally, using the result (3) and $\nabla_\beta f = \partial_\beta f$ the Laplacian can be written as

   $$\Box f = \nabla_\alpha (g^{\alpha\beta}\nabla_\beta f) = g^{-1/2}\partial_\alpha (g^{1/2} g^{\alpha\beta}\partial_\beta f) \ . \tag{5}$$

   (c) To calculate the Laplacian on the 2-sphere S^2, we just need the metric

   $$ds^2 = g_{ab} dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2 \tag{6}$$

   (with $x^a = (\theta, \phi)$). Then $\sqrt{g} = \sin\theta$, the non-zero components of the inverse metric are $g^{\theta\theta} = 1, g^{\phi\phi} = (\sin\theta)^{-2}$. Therefore (for once writing out everything in perhaps more detail than needed)

   $$\begin{aligned} \Delta_{S^2} &= \frac{1}{\sin\theta}\partial_a (\sin\theta g^{ab}\partial_b) \\ &= \frac{1}{\sin\theta}\partial_\theta (\sin\theta g^{\theta\theta}\partial_\theta) + \frac{1}{\sin\theta}\partial_\phi (\sin\theta g^{\phi\phi}\partial_\phi) \\ &= \frac{1}{\sin\theta}\partial_\theta (\sin\theta\partial_\theta) + \frac{1}{\sin\theta}\partial_\phi ((\sin\theta)^{-1}\partial_\phi) \\ &= \partial_\theta^2 + \cot\theta\partial_\theta + (\sin\theta)^{-2}\partial_\phi^2 \ . \end{aligned} \tag{7}$$

The Euclidean metric on $\mathbb{R}^3$ in spherical coordinates $\{x^\alpha\} = \{r, \theta, \phi\} = \{r, x^a\}$ is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{8}$$

with inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \tag{9}$$

(note that the angular components have an additional factor of r^{-2} relative to those of the metric on the 2-sphere) and determinant,

$$g = r^4\sin^2\theta \quad \Rightarrow \quad \sqrt{g} = r^2\sin\theta \tag{10}$$

It then follows that the Laplace operator on $\mathbb{R}^3$ in spherical coordinates is

$$\begin{aligned} \Delta &= \frac{1}{r^2\sin\theta}\partial_\alpha(r^2\sin\theta g^{\alpha\beta}\partial_\beta) \\ &= \frac{1}{r^2\sin\theta}\Big(\partial_r(r^2\sin\theta\partial_r) + \partial_a(r^2\sin\theta g^{ab}\partial_b)\Big) \\ &= \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2} \quad . \end{aligned} \tag{11}$$

Remark: This calculation evidently generalises to any dimension. Thus in spherical coordinates in which the Euclidean metric has the form

$$ds^2 = dr^2 + r^2 d\Omega_n^2 \tag{12}$$

(with $d\Omega_n^2$ the line element on the unit n-sphere), the Laplace operator on $\mathbb{R}^{n+1}$ takes the form

$$\Box = \partial_r^2 + \frac{n}{r}\partial_r + \frac{\Delta_{S^n}}{r^2} \tag{13}$$

2. STATIC SCHWARZSCHILD OBSERVERS IN EDDINGTON-FINKELSTEIN COORDINATES

In ingoing EF coordinates, the Schwarschild metric takes the form

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2 \tag{14}$$

where $v = t + r_*$ satisfies

$$dv = dt + dr_* = dt + dr/f(r) \quad \Rightarrow \quad \frac{\partial v}{\partial t} = 1 \quad , \quad \frac{\partial v}{\partial r} = f(r)^{-1} \quad . \tag{15}$$

A static observer has 4-velocity

$$(u^\alpha)_{EF} = (u^v, u^r, u^\theta, u^\phi) = (u^v, 0, 0, 0) \ , \tag{16}$$

with

$$g_{\alpha\beta}u^\alpha u^\beta = -f(r)(u^v)^2 = -1 \quad \Rightarrow \quad u^v = f(r)^{-1/2} \ . \tag{17}$$

Thus

$$(u^\alpha)_{EF} = (f(r)^{-1/2}, 0, 0, 0) \ . \tag{18}$$

This agrees with the result in SS coordinates, as could have also been deduced from the vectorial transformation behaviour (y^μ are now the SS coordinates)

$$(u^v)_{EF} = \frac{\partial v}{\partial y^\mu}(u^\mu)_{SS} = \frac{\partial v}{\partial t}(u^t)_{SS} = (u^t)_{SS} \ . \tag{19}$$

(a) Since $\dot{u}^\alpha = 0$ for a static oberver, the acceleration is

$$a^\alpha = \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = \Gamma^\alpha_{vv}(u^v)^2 = f(r)^{-1}\Gamma^\alpha_{vv} \ . \tag{20}$$

Even though Christoffel symbols are non-tensorial in general, they do transform in a simple way under the very simple coordinate transformation between SS and EF coordinates. Nevertheless it is more convenient (and in any case a good exercise) to just calculate the relevant Christoffel symbols directly in EF coordinates rather than transforming them from the result in SS coordinates.

As usual, we can calculate the Christoffel symbols either directly, or we can read them off from the Euler-Lagrange equations. As an exercise, you should do the calculation in both ways:

- Direct Calculation
  First of all, the Christoffel symbol $\Gamma_{\alpha vv}$ is rather obviously non-zero only for $\alpha = r$, and

$$\Gamma_{rvv} = -\tfrac{1}{2}g_{vv,r} = +\tfrac{1}{2}f'(r) = m/r^2 \ . \tag{21}$$

  To determine Γ^α_{vv} we need the components of the inverse metric. Because the metric is not diagonal, this requires a bit of care. In matrix form, the (v,r)-components of the metric and its inverse are

$$(g_{\alpha\beta}) = \begin{pmatrix} -f & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (g^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & f \end{pmatrix} \tag{22}$$

  Therefore, there are two non-vanishing Christoffel symbols Γ^α_{vv}, namely

$$\Gamma^v_{vv} = \Gamma_{rvv} = \tfrac{1}{2}f'(r) \quad , \quad \Gamma^r_{vv} = f(r)\Gamma_{rvv} = \tfrac{1}{2}f(r)f'(r) \ , \tag{23}$$

- From the Euler-Lagrange Equations
  From the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}f(r)\dot{v}^2 + \dot{r}\dot{v} + \ldots \tag{24}$$

we obtain the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{v}} - \frac{\partial\mathcal{L}}{\partial v} &= -f'(r)\dot{r}\dot{v} - f(r)\ddot{v} + \ddot{r} = 0 \\ \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} &= \ddot{v} + \tfrac{1}{2}f'(r)\dot{v}^2 + \ldots = 0 \end{aligned} \tag{25}$$

where the dots ... indicate terms that are irrelevant for the calculation of the Christoffel symbols $\Gamma^{\alpha}{}_{vv}$. From the 2nd equation one reads off

$$\Gamma^{v}{}_{vv} = \tfrac{1}{2}f'(r) \tag{26}$$

and using the 2nd equation to eliminate $\ddot{v}$ from the 1st equation, one finds

$$\ddot{r} + \tfrac{1}{2}f(r)f'(r)\dot{v}^2 - f'(r)\dot{r}\dot{v} + \ldots = 0 \quad \Rightarrow \quad \Gamma^{r}{}_{vv} = \tfrac{1}{2}f(r)f'(r) \tag{27}$$

This agrees completely with the result obtained above.

Either way one finds that the acceleration vector is

$$(a^{\alpha})_{EF} = (a^v = f(r)^{-1}m/r^2, a^r = m/r^2, 0, 0) \ . \tag{28}$$

Thus the (non-singular, "Newtonian") r-component agrees with that of the acceleration in SS coordinates, but in addition in EF coordinates there is a v-component which is singular as $r \to 2m$.

(b) Alternatively, and more quickly, since a^{α} is a vector, one can obtain this result by transforming the result in SS coordinates to EF coordinates,

$$(a^{\alpha})_{EF} = \frac{\partial x^{\alpha}}{\partial y^{\mu}}(a^{\mu})_{SS} = \frac{\partial x^{\alpha}}{\partial r}(a^{r})_{SS} \quad \Rightarrow \quad \begin{cases} (a^v)_{EF} = f(r)^{-1}m/r^2 \\ (a^r)_{EF} = m/r^2 \end{cases} \tag{29}$$

(c) Finally, the norm of the acceleration in EF coordinates is

$$g_{\alpha\beta}a^{\alpha}a^{\beta} = -f(r)(a^v)^2 + 2a^v a^r = f(r)^{-1}(m/r^2)^2 \ , \tag{30}$$

in complete agreement with the result in SS coordinates (as it should).