Solutions to Assignments 01

1. Free Relativistic Particle in Arbitrary Coordinates

   - From
   $$g_{\mu\nu} = \eta_{ab} J^a_\mu J^b_\nu \tag{1}$$
   one deduces
   $$g_{\mu\nu,\lambda} = \eta_{ab}(J^a_{\mu\lambda} J^b_\nu + J^a_\mu J^b_{\nu\lambda}) \tag{2}$$
   where
   $$J^a_{\mu\lambda} = \partial_\lambda J^a_\mu = \frac{\partial^2 \xi^a}{\partial x^\mu \partial x^\lambda} = J^a_{\lambda\mu} \quad . \tag{3}$$
   - Therefore one has
   $$\begin{aligned}\Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \\ &= \tfrac{1}{2}\eta_{ab}(J^a_{\mu\lambda}J^b_\nu + J^a_\mu J^b_{\nu\lambda} + J^a_{\mu\nu}J^b_\lambda + J^a_\mu J^b_{\lambda\nu} - J^a_{\nu\mu}J^b_\lambda - J^a_\nu J^b_{\lambda\mu}) \\ &= \eta_{ab} J^a_\mu J^b_{\nu\lambda} \quad ,\end{aligned} \tag{4}$$
   where the cancellations in passing to the last line arise from the symmetries $\eta_{ab} = \eta_{ba}$, $J^b_{\lambda\mu} = J^b_{\mu\lambda}$ etc.
   - Thus (writing out everything in detail),
   $$\begin{aligned}\Gamma^\mu{}_{\nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} = \eta^{cd} J^\mu_c J^\rho_d \eta_{ab} J^a_\rho J^b_{\nu\lambda} = \eta^{cd} J^\mu_c \delta^a_d \eta_{ab} J^b_{\nu\lambda} \\ &= \eta^{ca} J^\mu_c \eta_{ab} J^b_{\nu\lambda} = \delta^c_b J^\mu_c J^b_{\nu\lambda} = J^\mu_b J^b_{\nu\lambda} \quad ,\end{aligned} \tag{5}$$
   as was to be shown.

2. Geodesics

   (a) With the Lagrangian
   $$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{6}$$
   where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ one computes
   $$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2}\dot{x}^\rho \dot{x}^\nu \partial_\mu g_{\rho\nu} \tag{7}$$
   $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\rho\nu}\dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu}\dot{x}^\nu = g_{\rho\nu}\dot{x}^\rho \delta^\nu_\mu = g_{\rho\mu}\dot{x}^\rho \tag{8}$$
   $$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = g_{\rho\mu}\ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} = g_{\rho\mu}\ddot{x}^\rho + \frac{1}{2}\left(\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\rho g_{\nu\mu}\right) \tag{9}$$
   Thus the Euler-Lagrange equations become
   $$\Big[\text{E.-L.}\Big] = g_{\mu\rho}\ddot{x}^\rho + \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu}\right)\dot{x}^\nu \dot{x}^\rho \tag{10}$$
   $$= g_{\mu\rho}\ddot{x}^\rho + \Gamma_{\mu\nu\rho}\dot{x}^\rho \dot{x}^\nu = 0 \tag{11}$$

and they can be written in the usual (geodesic equation) form by multiplying by $g^{\lambda\mu}$ to move the index μ up:

$$g^{\lambda\mu}\left(g_{\mu\rho}\ddot{x}^{\rho}+\Gamma_{\mu\nu\rho}\dot{x}^{\rho}\dot{x}^{\nu}\right)=\ddot{x}^{\lambda}+\Gamma^{\lambda}_{\nu\rho}\dot{x}^{\rho}\dot{x}^{\nu}=0 \tag{12}$$

(b) First we compute :

$$\frac{d}{d\tau}\mathcal{L}=\frac{1}{2}\frac{d}{d\tau}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=\frac{1}{2}\left(2g_{\mu\nu}\ddot{x}^{\mu}\dot{x}^{\nu}+\ (\dot{x}^{\rho}\partial_{\rho}g_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu}\right) \tag{13}$$

From the definition of the Christoffel symbols $\Gamma_{\mu\nu\rho}$ one sees that if one symmetrises the 1st and 2nd index, 4 of the 6 terms cancel while 2 add up, leading to the useful identity

$$\partial_{\rho}g_{\mu\nu}=\Gamma_{\mu\nu\rho}+\Gamma_{\nu\mu\rho}\ \ . \tag{14}$$

Using this identity together with the fact that $x^{\mu}(\tau)$ is a solution to the geodesic equation, which means that we also have

$$\ddot{x^{\mu}}=-\Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} \tag{15}$$

leaves us with

$$\begin{aligned}\frac{d}{d\tau}\mathcal{L} &= \frac{1}{2}\left(-2g_{\mu\nu}\Gamma^{\mu}_{\lambda\rho}\dot{x}^{\lambda}\dot{x}^{\rho}\dot{x}^{\nu}+\dot{x}^{\rho}(\Gamma_{\mu\nu\rho}+\Gamma_{\nu\mu\rho})\dot{x}^{\mu}\dot{x}^{\nu}\right) &(16)\\ &= \frac{1}{2}\left(-2\Gamma_{\nu\lambda\rho}\dot{x}^{\lambda}\dot{x}^{\rho}\dot{x}^{\nu}+(\Gamma_{\mu\nu\rho}+\Gamma_{\nu\mu\rho})\dot{x}^{\mu}\dot{x}^{\nu}\dot{x}^{\rho}\right)=0 &(17)\end{aligned}$$

which is obviously zero if we relabel the indices.

(c) The metric on the 2-sphere is

$$ds^2=R^2\left(d\theta^2+\sin^2(\theta)d\phi^2\right) \tag{18}$$

so that in the (θ,ϕ) coordinates we have :

$$g_{\mu\nu}=R^2\begin{pmatrix}1 & 0\\ 0 & \sin(\theta)^2\end{pmatrix}\qquad\qquad g^{\mu\nu}=R^{-2}\begin{pmatrix}1 & 0\\ 0 & \sin(\theta)^{-2}\end{pmatrix} \tag{19}$$

Because $g_{\mu\nu}$ is diagonal and $g_{\mu\nu}=g_{\mu\nu}(\theta)$, the only non-vanishing contributions to the Christoffel symbols will come from terms involving $\partial_{\theta}g_{\phi\phi}$. Keeping this in mind, we first compute the Christoffel symbols with upper index θ,

$$\Gamma^{\theta}_{\nu\lambda}=\frac{1}{2}g^{\theta\theta}\left(\partial_{\lambda}g_{\theta\nu}+\partial_{\nu}g_{\theta\lambda}-\partial_{\theta}g_{\nu\lambda}\right)=-\frac{1}{2}g^{\theta\theta}\partial_{\theta}g_{\nu\lambda} \tag{20}$$

and we see that the only non-vanishing term with θ on the top is $\Gamma^{\theta}_{\phi\phi}$:

$$\Gamma^{\theta}_{\phi\phi}=-\frac{1}{2}g^{\theta\theta}\partial_{\theta}g_{\phi\phi}=-\sin(\theta)\cos(\theta) \tag{21}$$

Now if we choose ϕ to be on the top, we get

$$\Gamma^{\phi}_{\nu\lambda}=\frac{1}{2}g^{\phi\phi}\left(\partial_{\lambda}g_{\phi\nu}+\partial_{\nu}g_{\phi\lambda}-\partial_{\phi}g_{\nu\lambda}\right)=\frac{1}{2}g^{\phi\phi}\left(\partial_{\lambda}g_{\phi\nu}+\partial_{\nu}g_{\phi\lambda}\right) \tag{22}$$

so that the only non-vanishing terms with ϕ on the top are $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi}$:

$$\Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{23}$$

With these Christoffel symbols we can now write down the geodesic equation $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$ in the (θ, ϕ) coordinate. We find

$$\begin{cases} \ddot{\theta} + \Gamma^{\theta}_{\phi\phi}\dot{\phi}\dot{\phi} = \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}\dot{\phi} = 0 \qquad \mu = \theta \\ \\ \ddot{\phi} + 2\Gamma^{\phi}_{\theta\phi}\dot{\theta}\dot{\phi} = \ddot{\phi} + 2\dfrac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\dot{\phi} = 0 \qquad \mu = \phi \end{cases} \tag{24}$$

Using the Euler-Lagrange equations with $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \sin(\theta)^2\dot{\phi}^2)$, we get :

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dfrac{\partial \mathcal{L}}{\partial \theta} = \dfrac{1}{2}\left(\dfrac{d}{d\tau}2\dot{\theta} - 2\sin(\theta)\cos(\theta)\dot{\phi}^2\right) = 0 \\ \\ \dfrac{d}{d\tau}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} - \dfrac{\partial \mathcal{L}}{\partial \phi} = \dfrac{1}{2}\dfrac{d}{d\tau}\left(2\sin(\theta)^2\dot{\phi}\right) = \sin(\theta)^2\ddot{\phi} + 2\cos(\theta)\sin(\theta)\dot{\theta}\dot{\phi} = 0 \end{cases} \tag{25}$$

which are the same equations as in (31).

As discussed during the course, conversely one can simply use the Euler-Lagrange equations to read off all the non-zero components of the Christoffel symbol.

We can easily see that the great circles $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ on S^2 are solutions to the equations just found. It is indeed the case because for that particular solution ϕ is constant along a great circle, which implies $\ddot{\phi} = \dot{\phi} = 0$ and simultaneously $\theta(\tau) = \tau$ so that $\ddot{\theta}$ vanishes. By looking at equation (19) or (20) we can see that every curve with $\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0$ trivially satisfy both equations and therefore such curves are geodesics.