Solutions to Assignments 05

1. On the Klein-Gordon Field in a curved space-time

   The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 \right) \tag{1}$$

   and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{2}$$

   (a) Variation of the mass term evidently gives rise to $-\int \sqrt{g} d^4x m^2 \phi\delta\phi$, so we focus on the kinetic term. Thinking of the partial derivatives in the action as covariant derivatives, and using $\nabla_\alpha g_{\mu\nu} = 0$, $\nabla_\alpha \sqrt{g} = 0$, one can integrate by parts to one's heart's content as in Minkowski space to find

$$\delta S = -\int \sqrt{g} d^4x \; g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi = \int \sqrt{g} d^4x \; g^{\mu\nu}(\nabla_\mu\nabla_\nu\phi)\delta\phi \tag{3}$$

   Combining the variations of the kinetic and mass terms, one deduces the equation of motion $(\Box - m^2)\phi = 0$. If one works with the partial derivatives, one of course finds the same equation, but this time with the Laplace operator in the form $\sqrt{g}\Box = \partial_\alpha(\sqrt{g} g^{\alpha\beta}\partial_\beta)$.

   (b) To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu\phi = \nabla_\mu\phi$, the commutativity $\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi$ of the covariant derivative on scalars, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu\nabla_\mu\phi = m^2\phi$, then the result follows:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu(\partial_\mu\phi\partial_\nu\phi) + \nabla^\mu(g_{\mu\nu}L) \\ &= \nabla^\mu(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}\nabla_\nu(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2) \\ &= \partial_\nu\phi\nabla^\mu\partial_\mu\phi + \partial_\mu\phi\nabla^\mu\partial_\nu\phi - \partial_\lambda\phi\nabla_\nu\partial^\lambda\phi - m^2\phi\nabla_\nu\phi \\ &= \partial_\nu\phi \; m^2\phi + \partial_\mu\phi\nabla^\mu\nabla_\nu\phi - \partial_\lambda\phi\nabla^\lambda\nabla_\nu\phi - m^2\phi\nabla_\nu\phi \\ &= 0 \end{aligned} \tag{4}$$

   (c) The variation of the action with respect to the metric is

$$\delta S = \int d^4x \left(\delta(\sqrt{g})L + \sqrt{g}\delta L\right) = -\tfrac{1}{2}\int d^4x \sqrt{g} \left(g_{\mu\nu}L\delta g^{\mu\nu} - 2\delta L\right) \tag{5}$$

   (valid for any Lagrangian L). Using $\delta(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = (\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi$, one finds

$$\delta S = -\frac{1}{2}\int d^4x \sqrt{g}(g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi)\delta g^{\mu\nu} = -\frac{1}{2}\int d^4x \sqrt{g} T_{\mu\nu}\delta g^{\mu\nu} \tag{6}$$

   as claimed.

2. On the Maxwell Equations in Curved Space-Time The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \tag{7}$$

and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta^{\ \gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \tag{8}$$

(a) The variation of the action with respect to the gauge field is

$$\begin{aligned} \delta S &= -\int \sqrt{g} d^4x \; (\partial_\mu \delta A_\nu) F^{\mu\nu} \\ &= -\int d^4x \; \partial_\mu(\sqrt{g} \delta A_\nu F^{\mu\nu}) + \int d^4x \; \partial_\mu(\sqrt{g} F^{\mu\nu}) \delta A_\nu \end{aligned} \tag{9}$$

where in the first line we have used the fact that there are 4 identical contributions to the variation. Then we note that the first term in the second line is a boundary term and vanishes because the variation vanishes on the boundary. Now using $\sqrt{g} \nabla_\mu F^{\mu\nu} = \partial_\mu(\sqrt{g} F^{\mu\nu})$ we are left with

$$\delta S = \int \sqrt{g} d^4x \; (\nabla_\mu F^{\mu\nu}) \delta A_\nu \tag{10}$$

which gives the equations of motion.

(b) We compute :

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu (F^\mu_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) \\ &= (\nabla_\mu F^\mu_{\ \lambda}) F^{\nu\lambda} + F^\mu_{\ \lambda} \nabla_\mu F^{\nu\lambda} - \tfrac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\ &= -J_\lambda F^{\nu\lambda} + F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2} \nabla^\nu F^{\mu\lambda} \right) \\ &= J_\lambda F^{\lambda\nu} + \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right) \end{aligned} \tag{11}$$

Then we rewrite the term $\frac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda}$ in a different way by relabeling the indices and using the anti-symmetry of $F_{\mu\nu}$:

$$\tfrac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda} = \tfrac{1}{2} F_{\lambda\mu} \nabla^\lambda F^{\nu\mu} = -\tfrac{1}{2} F_{\mu\lambda} \nabla^\lambda F^{\nu\mu} \tag{12}$$

so that we can now use $\nabla_{[\lambda} F_{\mu\nu]} = 0$ to have at the end :

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= J_\lambda F^{\lambda\nu} - \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) \\ &= J_\lambda F^{\lambda\nu} \end{aligned} \tag{13}$$

(c) For the metric variation of the action, we can also use the general formula (5). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho}) = 2(\delta g^{\mu\lambda}) g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} = 2(\delta g^{\mu\nu}) g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} = 2(\delta g^{\mu\nu}) F_{\mu\lambda} F_\nu^{\ \lambda} \tag{14}$$

and therefore $-2\delta L = (\delta g^{\mu\nu}) F_{\mu\lambda} F_\nu^{\ \lambda}$. Putting the pieces together one gets (8).