Solutions to Assignments 04

1. Tensor Analysis II: the Covariant Derivative

   (a) Consider the scalar $A_\nu V^\nu$ and take its covariant derivative. Since it is a scalar, its covariant and partial derivatives agree, and since both satisfy the Leibniz rule one has

$$\begin{aligned}\nabla_\mu(A_\nu V^\nu) &= \partial_\mu(A_\nu V^\nu) = A_\nu\partial_\mu V^\nu + V^\nu\partial_\mu A_\nu \\ &= A_\nu\nabla_\mu V^\nu + V^\nu\nabla_\mu A_\nu\end{aligned} \tag{1}$$

   This implies

$$\begin{aligned}V^\nu\nabla_\mu A_\nu &= V^\nu\partial_\mu A_\nu + A_\nu\partial_\mu V^\nu - A_\nu\nabla_\mu V^\nu \\ &= V^\nu\partial_\mu A_\nu + A_\nu\partial_\mu V^\nu - A_\nu(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho) \\ &= V^\nu\partial_\mu A_\nu - A_\nu\Gamma^\nu_{\mu\rho}V^\rho = V^\nu\partial_\mu A_\nu - A_\lambda\Gamma^\lambda_{\mu\nu}V^\nu \\ \Rightarrow \qquad \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu}A_\lambda\end{aligned} \tag{2}$$

   the last implication following because this has to be true for any V^ν.

   (b) Since $A_{\nu'} = J^\nu_{\nu'}A_\nu$ and $\partial_{\mu'} = J^\mu_{\mu'}\partial_\mu$, one has

$$\begin{aligned}\partial_\mu A_\nu \to \partial_{\mu'}A_{\nu'} &= J^\mu_{\mu'}\partial_\mu(J^\nu_{\nu'}A_\nu) \\ &= J^\mu_{\mu'}J^\nu_{\nu'}\partial_\mu A_\nu + A_\nu J^\mu_{\mu'}\partial_\mu J^\nu_{\nu'} \\ &= J^\mu_{\mu'}J^\nu_{\nu'}\partial_\mu A_\nu + A_\nu J^\nu_{\mu'\nu'} \ .\end{aligned} \tag{3}$$

   Thus this is not a tensor, but since the last term is symmetric in the free indices,

$$J^\nu_{\mu'\nu'} = \frac{\partial^2 x^\nu}{\partial y^{\mu'}\partial y^{\nu'}} = J^\nu_{\nu'\mu'} \tag{4}$$

   (partial derivatives commute), it drops out when one takes the antisymmetric part, i.e. the curl,

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} = J^\mu_{\mu'}J^\nu_{\nu'}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{5}$$

   Because the Christoffel symbols are symmetric in their lower indices, they always drop out of the anti-symmetrised derivatives of anti-symmetric covariant tensors. In the present (simplest) case of covectors, one has

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu}A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu}A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{6}$$

   (c) • Argument by direct calculation:

$$\begin{aligned}\nabla_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu}g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda}g_{\nu\rho} \\ &= \partial_\mu g_{\nu\lambda} - \Gamma_{\lambda\mu\nu} - \Gamma_{\nu\mu\lambda} = 0\end{aligned} \tag{7}$$

   from the explicit form of the Christoffel symbols.

- Alternative argument: Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial coordinate system at x. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.

2. Stationary and Freely Falling Schwarzschild Observers

(a) The observer is sitting at fixed radius and angles, therefore his worldline 4-velocity is of the form

$$\frac{dx^\mu}{d\tau} = u^\mu = (u^t, 0, 0, 0) \ . \tag{8}$$

The proper time normalisation condition implies

$$u^\mu u_\mu = -1 \quad \Rightarrow \quad u^t = \frac{1}{\sqrt{1-\frac{2m}{r}}} \tag{9}$$

(we have chosen $u^t > 0$ because the oberver evolves forward in time, $\dot{t} > 0$). The acceleration is then

$$\begin{aligned} a^\mu = \nabla_\tau u^\mu &= u^\rho \nabla_\rho u^\mu \\ &= u^t \partial_t u^\mu + u^t \Gamma^\mu_{tt} u^t \\ &= \Gamma^\mu_{tt} \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} g^{\mu\rho} \partial_\rho g_{tt} \frac{1}{1-\frac{2m}{r}} \\ \text{for } \mu \neq r \quad &= 0 \\ \text{for } \mu = r \quad &= \frac{1}{2} g^{rr} \partial_r (1 - \frac{2m}{r}) \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} \partial_r \frac{2m}{r} = \frac{m}{r^2} \end{aligned} \tag{10}$$

and therefore the norm of the acceleration is

$$\begin{aligned} g_{\mu\nu} a^\mu a^\nu &= g_{rr} a^r a^r \\ &= \frac{1}{1-\frac{2m}{r}} \frac{m^2}{r^4} \ . \end{aligned} \tag{11}$$

Note that this approaches the Newtonian value $(m/r^2)^2$ for $r \to \infty$, while the required acceleration to keep the stationary observer at rest diverges as $r \to 2m$.

(b) For zero angular momentum, and with $\dot{r}_{r=R} = 0$ the effective potential equation reduces to

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} \quad \Rightarrow \quad \dot{r}^2 = \frac{2m}{r} - \frac{2m}{R} \ , \tag{12}$$

which integrates to

$$\tau_{R\to r_1} = -(2m)^{-1/2} \int_R^{r_1} dr \, \left(\frac{Rr}{R-r} \right)^{1/2} \quad . \tag{13}$$

This integral can be calculated in closed form, e.g. via the change of variables

$$\frac{r}{R} = \sin^2 \alpha \qquad \alpha_1 \leq \alpha \leq \frac{\pi}{2} \quad , \tag{14}$$

leading to

$$\tau_{R\to r_1} = 2 \left(\frac{R^3}{2m} \right)^{1/2} \int_{\alpha_1}^{\pi/2} d\alpha \, \sin^2 \alpha = \left(\frac{R^3}{2m} \right)^{1/2} \left[\alpha - \tfrac{1}{2} \sin 2\alpha \right]_{\alpha_1}^{\pi/2} \quad . \tag{15}$$

For $r_1 \to 0 \Leftrightarrow \alpha_1 \to 0$ one obtains

$$\tau_{R\to 0} = \left(\frac{R^3}{2m} \right)^{1/2} (\pi/2) = \pi \left(\frac{R^3}{8m} \right)^{1/2} \tag{16}$$

R and $r_S = 2m$ have dimensions of length, thus the quantity above also has dimensions of length, so what we have actually calculated is $c\tau$, not τ. To obtain proper time, we thus need to divide by c. Using the approximate values

$$(R)_{\text{sun}} \approx 7 \times 10^{10} \mathsf{cm} \quad (2m)_{\text{sun}} \approx 3 \times 10^{5} \mathsf{cm} \quad c \approx 3 \times 10^{10} \mathsf{cm\ s}^{-1} \tag{17}$$

one finds $\tau_{R\to 0} \approx 2 \times 10^3 \mathsf{s}$, which is roughly 30 minutes.