

SOLUTIONS TO ASSIGNMENTS 05

1. TENSOR ANALYSIS III: THE COVARIANT DIVERGENCE

(a) First we compute $\Gamma_{\mu\lambda}^{\mu}$ with the definition :

$$\begin{aligned}\Gamma_{\mu\lambda}^{\mu} &= \frac{1}{2}g^{\mu\rho}(\partial_{\mu}g_{\rho\lambda} + \partial_{\lambda}g_{\mu\rho} - \partial_{\rho}g_{\mu\lambda}) \\ &= \frac{1}{2}(\partial^{\rho}g_{\rho\lambda} + g^{\mu\rho}\partial_{\lambda}g_{\mu\rho} - \partial^{\mu}g_{\mu\lambda}) \\ &= \frac{1}{2}g^{\mu\rho}\partial_{\lambda}g_{\mu\rho}\end{aligned}\tag{1}$$

Then, we use the relation $g^{-1}\partial_{\lambda}g = g^{\mu\nu}\partial_{\lambda}g_{\mu\nu}$ to find that :

$$\begin{aligned}\Gamma_{\mu\lambda}^{\mu} &= \frac{1}{2}g^{\mu\rho}\partial_{\lambda}g_{\mu\rho} \\ &= \frac{1}{2}g^{-1}\partial_{\lambda}g \\ &= g^{-1/2}\partial_{\lambda}g^{+1/2}\end{aligned}\tag{2}$$

where in the last equality we used the fact that : $\partial_{\lambda}g^{+1/2} = \frac{1}{2}g^{-1/2}\partial_{\lambda}g$.

(b) We can now compute the covariant divergence :

$$\begin{aligned}\nabla_{\mu}J^{\mu} &= \partial_{\mu}J^{\mu} + \Gamma_{\mu\rho}^{\mu}J^{\rho} \\ &= \partial_{\mu}J^{\mu} + J^{\rho}g^{-1/2}\partial_{\rho}g^{+1/2} \\ &= g^{-1/2}\partial_{\mu}(g^{1/2}J^{\mu})\end{aligned}\tag{3}$$

and

$$\begin{aligned}\nabla_{\mu}F^{\mu\nu} &= \partial_{\mu}F^{\mu\nu} + \Gamma_{\mu\rho}^{\mu}F^{\rho\nu} + \Gamma_{\mu\rho}^{\nu}F^{\mu\rho} \\ &= \partial_{\mu}F^{\mu\nu} + \Gamma_{\mu\rho}^{\mu}F^{\rho\nu} \\ &= \partial_{\mu}F^{\mu\nu} + F^{\rho\nu}g^{-1/2}\partial_{\rho}g^{+1/2} \\ &= g^{-1/2}\partial_{\mu}(g^{1/2}F^{\mu\nu})\end{aligned}\tag{4}$$

where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma_{\mu\rho}^{\nu}$). More precisely, if we rewrite $\Gamma_{\mu\rho}^{\nu} = \frac{1}{2}(\Gamma_{\mu\rho}^{\nu} + \Gamma_{\rho\mu}^{\nu})$ and $F^{\mu\rho} = \frac{1}{2}(F^{\mu\rho} - F^{\rho\mu})$, then $\Gamma_{\mu\rho}^{\nu}F^{\mu\rho}$ contains 4 terms and relabelling two of them by the exchange of the indices $\mu \leftrightarrow \rho$ we see that everything vanishes.

(c) To calculate the Laplacian, we just need the metric,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}\tag{5}$$

its inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \quad (6)$$

and its determinant,

$$g = r^4 \sin^2 \theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin \theta \quad (7)$$

Then one calculates

$$\begin{aligned} \square\Phi &= \frac{1}{r^2 \sin \theta} \partial_\alpha (r^2 \sin \theta g^{\alpha\beta} \partial_\beta \Phi) \\ &= \frac{1}{r^2 \sin \theta} (\partial_r (r^2 \sin \theta \partial_r \Phi) + \partial_\theta (\sin \theta \partial_\theta \Phi) + \partial_\phi ((\sin \theta)^{-1} \partial_\phi \Phi)) \\ &= r^{-2} \partial_r (r^2 \partial_r \Phi) + r^{-2} ((\sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta \Phi) + (\sin \theta)^{-2} \partial_\phi^2 \Phi) \end{aligned} \quad (8)$$

This can now be rewritten in many ways, e.g. as

$$\square = \partial_r^2 + \frac{2}{r} \partial_r + \frac{\Delta_{S^2}}{r^2} \quad (9)$$

with

$$\Delta_{S^2} = \frac{1}{\sin \theta} \partial_a (\sin \theta g^{ab} \partial_b) \quad (10)$$

($x^a = (\theta, \phi)$) the Laplace operator on the unit 2-sphere.

2. ORBIT EQUATION FOR GENERAL STATIC SPHERICALLY SYMMETRIC METRICS

(a) First we extract the constants of motion from the Lagrangian :

$$\begin{aligned} \mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{\epsilon}{2} &= -\frac{1}{2} A(r) \dot{t}^2 + \frac{1}{2} B(r) \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{1}{2} r^2 \sin(\theta)^2 \dot{\phi}^2 \\ &\Rightarrow -\frac{\partial \mathcal{L}}{\partial t} = E = A(r) \dot{t} \end{aligned} \quad (11)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L = r^2 \sin(\theta)^2 \dot{\phi} \quad (12)$$

$$\text{and we fix } \theta = \theta_0 = \frac{\pi}{2} \quad \Rightarrow \quad \mathcal{L} = -\frac{E^2}{2A(r)} + \frac{B(r)}{2} \dot{r}^2 + \frac{L^2}{2r^2} = \frac{\epsilon}{2} \quad (13)$$

We have $r = r(\tau)$ but we want to express r as a function of ϕ , as we do so this implies that $\dot{r} = r' \dot{\phi} = \frac{L r'}{r^2}$ and we get with $r = r(\phi)$ and $r' = \frac{dr}{d\phi}$:

$$-\frac{E^2}{L^2 A(r)} + B(r) \frac{r'^2}{r^4} + \frac{1}{r^2} = \frac{\epsilon}{L^2} \quad (14)$$

Finally, we change variable from r to $u = 1/r$ ($\Rightarrow r' = -\frac{u'}{u^2}$) and (4) becomes :

$$\tilde{B}(u) u'^2 + u^2 = \frac{\epsilon}{L^2} + \frac{E^2}{L^2 \tilde{A}(u)} \quad (15)$$

where $\tilde{B}(u) = B(r(u)) = B(\frac{1}{u})$ and the same for $\tilde{A}(u)$.

(b) For $\epsilon = 0$ one has

$$\tilde{B}(u)u'^2 + u^2 = \frac{E^2}{L^2\tilde{A}(u)} \quad (16)$$

Thus at a truning point one has

$$u_m^2 = \frac{E^2}{L^2\tilde{A}(u_m)} \quad (17)$$

and one can trade the parameter E^2/L^2 for u_m ,

$$\tilde{B}(u)u'^2 + u^2 = u_m^2 \frac{\tilde{A}(u_m)}{\tilde{A}(u)} \Rightarrow \frac{d\phi}{du} = \pm\tilde{B}(u)^{1/2} \left[u_m^2 \frac{\tilde{A}(u_m)}{\tilde{A}(u)} - u^2 \right]^{-1/2} \quad (18)$$

(c) For $A = B = 1$ one gets (choosing wlog the + sign in the above equation),

$$\frac{d\phi}{du} = [u_m^2 - u^2]^{-1/2} \quad (19)$$

This elementary differential equation is solved by

$$u(\phi) = u_m \sin(\phi - \phi_0) \Rightarrow r \sin(\phi - \phi_0) = u_m^{-1} \quad (20)$$

which is the equation of a straight line in \mathbb{R}^2 ($ax+by = c$) in polar coordinates.