Solutions to Assignments 04

1. Tensor Analysis II: the Covariant Derivative

(a) Consider the scalar $A_\nu V^\nu$ and take its covariant derivative. Since it is a scalar, its covariant and partial derivatives agree, and since both satisfy the Leibniz rule one has

$$\begin{aligned}\nabla_\mu(A_\nu V^\nu) &= \partial_\mu(A_\nu V^\nu) = A_\nu \partial_\mu V^\nu + V^\nu \partial_\mu A_\nu \\ &= A_\nu \nabla_\mu V^\nu + V^\nu \nabla_\mu A_\nu\end{aligned} \tag{1}$$

This implies

$$\begin{aligned}V^\nu \nabla_\mu A_\nu &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu \nabla_\mu V^\nu \\ &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) \\ &= V^\nu \partial_\mu A_\nu - A_\nu \Gamma^\nu_{\mu\rho} V^\rho = V^\nu \partial_\mu A_\nu - A_\lambda \Gamma^\lambda_{\mu\nu} V^\nu \\ \Rightarrow \qquad \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda\end{aligned} \tag{2}$$

the last implication following because this has to be true for any V^ν.

(b) Since $A_{\nu'} = J^\nu_{\nu'} A_\nu$ and $\partial_{\mu'} = J^\mu_{\mu'} \partial_\mu$, one has

$$\begin{aligned}\partial_\mu A_\nu \to \partial_{\mu'} A_{\nu'} &= J^\mu_{\mu'} \partial_\mu (J^\nu_{\nu'} A_\nu) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\mu_{\mu'} \partial_\mu J^\nu_{\nu'} \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\nu_{\mu'\nu'} \ .\end{aligned} \tag{3}$$

Thus this is not a tensor, but since the last term is symmetric in the free indices,

$$J^\nu_{\mu'\nu'} = \frac{\partial^2 x^\nu}{\partial y^{\mu'} \partial y^{\nu'}} = J^\nu_{\nu'\mu'} \tag{4}$$

(partial derivatives commute), it drops out when one takes the antisymmetric part, i.e. the curl,

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} = J^\mu_{\mu'} J^\nu_{\nu'} (\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{5}$$

Because the Christoffel symbols are symmetric in their lower indices, they always drop out of the anti-symmetrised derivatives of anti-symmetric covariant tensors. In the present (simplest) case of covectors, one has

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{6}$$

(c)
- Argument by direct calculation: see lecture notes equation (4.30).
- Alternative argument: Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial coordinate system at x. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.

2. Radial Fall & the repulsive Reissner-Nordstrøm Core

(a) For this metric, the function $\phi(r)$ (the Newtonian-like potential) is $\phi(r) = -m/r + q^2/2r^2$. For radially infalling particles, the angular momentum is zero, and thus the effective potential is, according to exercise 03.2

$$V_{eff}(r) = \phi(r)(-\epsilon + L^2/r^2) + L^2/2r^2 \rightarrow \phi(r) = -\frac{m}{r} + \frac{q^2}{2r^2} \quad . \tag{7}$$

For $q \neq 0$, the 2nd term dominates over the 1st for $r \rightarrow 0$, and therefore a radially infalling particle never reaches $r = 0$. In fact, this effective potential is equivalent to the effective Newtonian potential for a particle with angular momentum $L = q$. By contrast, radially infalling particles either in the Newtonian case or in the Schwarzschild geometry experience no angular momentum barrier and thus reach $r = 0$ (in finite proper time).

(b) From the effective potential equation

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = E_{eff} \tag{8}$$

one deduces that a particle at rest ($\dot{r} = 0$) at infinity ($V_{eff}(r = \infty) = 0$) has $E_{eff} = 0$ (or $E^2 = 1$). The other turning point r_{min} of the trajectory, i.e. with $\dot{r}_{min} = 0$, is then given by the non-trivial solution of $V_{eff}(r) = 0$, namely

$$-\frac{m}{r_{min}} + \frac{q^2}{2r_{min}^2} = 0 \quad \Rightarrow \quad r_{min} = \frac{q^2}{2m} \tag{9}$$

The two roots of $f(r)$ (whose significance will be explained later on in the course) are

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} = 0 \quad \Rightarrow \quad r = r_\pm = m \pm \sqrt{m^2 - q^2} \quad . \tag{10}$$

For $q = 0$, r_+ reduces to the critical radius $2m$ of the Schwarzschild metric, and the condition $m^2 > q^2$ ensures that we have two real roots. It is now a simple matter to see that $r_{min} < r_-$. First of all one obviously has

$$\frac{q^2}{2m} < m - \sqrt{m^2 - q^2} \quad \Leftrightarrow \quad \sqrt{m^2 - q^2} < m - q^2/2m \tag{11}$$

Next, the term on the right hand side is indeed positive (from $m^2 > q^2$), so we can now square both sides to find

$$m^2 - q^2 < m^2 - q^2 + q^4/4m^2 \tag{12}$$

which is obviously a true statement. It is physically plausible (and also easy to check) that for $E^2 > 1$, i.e. for a particle with non-zero initial velocity, the minimal radius is smaller than for $E^2 = 1$.