Solutions to Assignments 03

1. Tensor analysis I: Tensor Algebra

   (a) Under a coordinate transformation $x^\mu \to x'^\alpha = y^\alpha(x)$ a scalar (function) $f(x)$ transforms as $f \to f'$ with $f'(y(x)) = f(x)$, while according to the chain rule the partial derivatives transform with the Jacobian as

   $$\partial_\alpha \equiv \partial_{y^\alpha} = J^\mu_\alpha \partial_\mu \quad . \tag{1}$$

   Thus

   $$\partial_\alpha f'(y) = J^\mu_\alpha \partial_\mu f(x) \tag{2}$$

   transforms as a covector.

   (b) By definition of a tensor we have (writing $A_{\alpha\beta}$ instead of $A'_{\alpha\beta}$ for simplicity, and suppressing the argument)

   $$A_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta A_{\mu\nu} \quad , \quad B^\beta = J^\beta_\rho B^\rho \quad , \tag{3}$$

   and therefore

   $$A_{\alpha\beta}B^\beta = J^\mu_\alpha J^\nu_\beta J^\beta_\rho A_{\mu\nu}B^\rho = J^\mu_\alpha \delta^\nu_\rho A_{\mu\nu}B^\rho = J^\mu_\alpha A_{\mu\nu}B^\nu \tag{4}$$

   is a covector. The same kind of argument now shows that

   $$A_{\alpha\beta}B^\alpha B^\beta = J^\alpha_\rho B^\rho J^\mu_\alpha A_{\mu\nu}B^\nu = \delta^\mu_\rho B^\rho A_{\mu\nu}B^\nu = B^\mu A_{\mu\nu}B^\nu \tag{5}$$

   is a scalar.

   (c) The invariance of $V(x)$ under coordinate transformations follows from the fact that partial derivatives are covectors and that they are contracted with a vector to form the field $V(x)$,

   $$V^\alpha \partial_\alpha = J^\alpha_\mu J^\nu_\alpha V^\mu \partial_\nu = \delta^\nu_\mu V^\mu \partial_\nu = V^\mu \partial_\mu \quad . \tag{6}$$

   Likewise for a covector:

   $$dy^\alpha = J^\alpha_\nu dx^\nu \quad \Rightarrow \quad A_\alpha dy^\alpha = J^\mu_\alpha J^\alpha_\nu A_\mu dx^\nu = A_\mu dx^\mu \quad . \tag{7}$$

2. The Effective Geodesic Potential

   Starting with the metric

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{8}$$

   one implements the following steps:

- the Lagrangian $\mathcal{L}$ is conserved,

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = \epsilon \tag{9}$$

  where $\epsilon = -1, 0$ for massive (massless) particles.

- by spherical symmetry, angular momentum is conserved, thus the motion is planar, and one can choosse the coordinates such that this motion takes place in the equatorial plane $\theta = \pi/2$, $\dot{\theta} = 0$, leading to

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon \quad . \tag{10}$$

- rotational and time-translational symmetry lead to the conserved quantities

$$E = f(r)\dot{t} \qquad L = r^2\dot{\phi} \tag{11}$$

  (energy and angular momentum), and using these equations to eliminate $\dot{t}$ and $\dot{\phi}$ from the Lagrangian, one finds

$$-E^2 f(r)^{-1} + f(r)^{-1}\dot{r}^2 + L^2/r^2 = \epsilon \quad . \tag{12}$$

  Multiplying by $f(r)$ and rearranging, this gives

$$\dot{r}^2 + f(r)L^2/r^2 - \epsilon f(r) = E^2 \quad . \tag{13}$$

- This already has the desired form of an effective Newtonian potential equation, but it is typically more useful to separate the constant (asymptotically Minkowski) part of $f(r)$ from the rest. Thus, with $f(r) = 1 + 2\phi(r)$ one has

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = E_{eff} \tag{14}$$

  where

$$V_{eff}(r) \equiv V(r) + L^2/2r^2 = \phi(r)(-\epsilon + L^2/r^2) + L^2/2r^2 \tag{15}$$

  and

$$E_{eff} = (E^2 + \epsilon)/2 \tag{16}$$