Solutions to Assignments 09 15.12.09

1. Properties of the Riemann Curvature Tensor

   (a) We show that the forth symmetry follows from (I),(II) and (III):

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -(R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta}) = R_{\gamma\alpha\delta\beta} + R_{\delta\alpha\beta\gamma} \\ &= -(R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\alpha\delta}) - (R_{\delta\beta\gamma\alpha} + R_{\delta\gamma\alpha\beta}) \\ &= 2R_{\gamma\delta\alpha\beta} + R_{\beta\gamma\alpha\delta} + R_{\beta\delta\gamma\alpha} \\ &= 2R_{\gamma\delta\alpha\beta} - R_{\beta\alpha\delta\gamma} = 2R_{\gamma\delta\alpha\beta} - R_{\alpha\beta\gamma\delta} \\ &\Rightarrow \ R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \end{aligned} \tag{1}$$

   (b) From (a) we directly deduce the symmetry of the Ricci tensor :

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = R_{\rho\nu}{}^{\rho}{}_{\mu} = R^{\rho}{}_{\nu\rho\mu} = R_{\nu\mu} \tag{2}$$

   (c) Writing $\circlearrowleft$ for the cyclic permutations in (α, β, γ) and then using the third symmetry : $R^{\rho}{}_{\alpha\beta\gamma} + \circlearrowleft = 0$, we have :

$$\begin{aligned} [\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft \quad &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma}V^\rho) - R^{\lambda}{}_{\rho\beta\gamma}\nabla_\alpha V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\ &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\ &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + \circlearrowleft \\ &= g^{\lambda\mu}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] V^\nu = 0 \end{aligned} \tag{3}$$

   which gives the desired result.

   (d) Contracting the Bianchi identity over the indices (μ, β) and (ν, α) one finds :

$$\begin{aligned} g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] \quad &= g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}\right] \\ &= \nabla_\alpha R^{\alpha\beta}{}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}{}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}{}_{\alpha\beta} \\ &= -\nabla_\alpha R^{\alpha}{}_{\gamma} - \nabla_\beta R^{\beta}{}_{\gamma} + \nabla_\gamma R \\ &= -\nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0 \end{aligned} \tag{4}$$

   And defining the *Einstein tensor* as $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$, we see that the contracted Bianchi identity (4) is equivalent to $\nabla^\alpha G_{\alpha\beta} = 0$ because :

$$\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R) = \frac{1}{2}\nabla_\alpha(2R^{\alpha}{}_{\beta} - g^{\alpha}{}_{\beta}R) \tag{5}$$

   so that $\nabla^\alpha G_{\alpha\beta} = 0 \ \Leftrightarrow \ \nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0$ where we have use the fact that $g^{\alpha}{}_{\beta} = g^{\alpha\lambda}g_{\lambda\beta} = \delta^{\alpha}{}_{\beta}$ simply because $g^{\alpha\lambda}$ is the inverse of $g_{\alpha\lambda}$.