1. ON THE KLEIN-GORDON FIELD IN A CURVED SPACE-TIME

   In a curved space-time the action for a free massive scalar field is :

$$S = \int d^4x\sqrt{-g}L = -\frac{1}{2}\int d^4x\sqrt{-g}(\ g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) \tag{1}$$

   We can find the energy-momentum tensor by computing the variation of the action with respect to the metric :

$$\delta S = \int d^4x\delta(\sqrt{-g})L + \sqrt{-g}\delta L \tag{2}$$

   Using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ one finds :

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}(g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi)\delta g^{\mu\nu} \tag{3}$$

   and the energy-momentum tensor is given by :

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi \tag{4}$$

   To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu\phi = \nabla_\mu\phi$, the commutativity $\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi$ of the covariant derivative on scalar, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu\nabla_\mu\phi = m^2\phi$, then the result follows:

$$\begin{aligned}
\nabla^\mu T_{\mu\nu} &= \nabla^\mu\partial_\mu\phi\partial_\nu\phi + \nabla^\mu g_{\mu\nu}L \\
&= \nabla^\mu\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\nabla_\nu(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2) \\
&= \partial_\nu\phi\nabla^\mu\partial_\mu\phi + \partial_\mu\phi\nabla^\mu\partial_\nu\phi - \partial_\lambda\phi\nabla_\nu\partial^\lambda\phi - m^2\phi\nabla_\nu\phi \\
&= \partial_\nu\phi\ m^2\phi + \partial_\mu\phi\nabla^\mu\nabla_\nu\phi - \partial_\lambda\phi\nabla^\lambda\nabla_\nu\phi - m^2\phi\nabla_\nu\phi \\
&= 0
\end{aligned} \tag{5}$$

2. ON THE MAXWELL EQUATIONS IN CURVED SPACE-TIME

   (a) We start from the action :

$$S = \tfrac{1}{4}\int \sqrt{g}d^4x\ g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = \tfrac{1}{4}\int \sqrt{g}d^4x\ F_{\mu\nu}F^{\mu\nu} \tag{6}$$

   and compute the variation with respect to the gauge field :

$$\begin{aligned}
\delta S &= \int \sqrt{g}d^4x\ (\partial_\mu\delta A_\nu)F^{\mu\nu} \\
&= \int d^4x\ \partial_\mu(\sqrt{g}\delta A_\nu F^{\mu\nu}) - \int d^4x\ \partial_\mu(\sqrt{g}F^{\mu\nu})\delta A_\nu
\end{aligned} \tag{7}$$

where in the first line we have used the fact that there are 4 identical contributions to the variation. Then we note that the first term in the second line is a boundary term and vanishes because the variation vanishes on the boundary. Now using $\sqrt{g}\nabla_\mu F^{\mu\nu} = \partial_\mu(\sqrt{g}F^{\mu\nu})$ we're left with :

$$\delta S = -\int \sqrt{g} d^4x \, (\nabla_\mu F^{\mu\nu})\delta A_\nu \tag{8}$$

witch gives the equations of motion.

(b) We compute :

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu (F^\mu_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) \\
&= (\nabla_\mu F^\mu_{\ \lambda}) F^{\nu\lambda} + F^\mu_{\ \lambda} \nabla_\mu F^{\nu\lambda} - \tfrac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\
&= -J_\lambda F^{\nu\lambda} + F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2} \nabla^\nu F^{\mu\lambda} \right) \\
&= J_\lambda F^{\lambda\nu} + \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right)
\end{aligned} \tag{9}$$

Then we rewrite the term $\frac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda}$ in a different way by relabeling the indices and using the anti-symmetry of $F_{\mu\nu}$:

$$\tfrac{1}{2} F_{\mu\lambda} \nabla^\mu F^{\nu\lambda} = \tfrac{1}{2} F_{\lambda\mu} \nabla^\lambda F^{\nu\mu} = -\tfrac{1}{2} F_{\mu\lambda} \nabla^\lambda F^{\nu\mu} \tag{10}$$

so that we can now use $\nabla_{[\lambda} F_{\mu\nu]} = 0$ to have at the end :

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= J_\lambda F^{\lambda\nu} - \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) \\
&= J_\lambda F^{\lambda\nu}
\end{aligned} \tag{11}$$

3. Tensor Analysis V : Parallel Transport

(a) By computing :

$$\frac{D\dot{x}^\mu}{D\tau} = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = \dot{x}^\nu \partial_\nu \dot{x}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\rho} \dot{x}^\rho = \ddot{x}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\rho} \dot{x}^\rho \tag{12}$$

we see that the tangent vector $\dot{x}^\mu$ is parallel transported along the curve $x^\mu(\tau)$ precisely when the curve is a geodesic :

$$\frac{D\dot{x}^\mu}{D\tau} = 0 \qquad \Leftrightarrow \qquad \ddot{x}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\rho} \dot{x}^\rho = 0 \tag{13}$$

(b) The length square of V^μ is $L^2 = g_{\mu\nu} V^\mu V^\nu$, and if V^μ is parallel transported along the curve this means $\frac{DV^\mu}{D\tau} = 0$, so we have :

$$\frac{DL^2}{D\tau} = \frac{D}{D\tau} g_{\mu\nu} V^\mu V^\nu = V^\mu V^\nu \frac{D}{D\tau} g_{\mu\nu} + 2V_\mu \frac{DV^\mu}{D\tau} = 0 \tag{14}$$

because $\nabla_\rho g_{\mu\nu} = 0$, and if L^2 is constant along the curve so is L.

(c) If V^μ is a parallel transported vector :

$$\frac{DV^\mu}{D\tau} = 0 \tag{15}$$

And because $g_{\mu\nu}\dot{x}^\mu V^\nu$ is a scalar, we also have :

$$\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\mu V^\nu) = \frac{D}{D\tau}(g_{\mu\nu}\dot{x}^\mu V^\nu) = V_\mu \frac{D\dot{x}^\mu}{D\tau} + \dot{x}_\mu \frac{DV^\mu}{D\tau} = 0 \tag{16}$$

if $x^\mu(\tau)$ is a geodesic, see (14).