

1. RINDLER COORDINATES AND THE SCHWARZSCHILD GEOMETRY NEAR $r = r_S$

- (a) We first look at curves $x^\mu(\tau)$ given by $x^0 = \rho_0 \sinh(\eta)$ and $x^1 = \rho_0 \cosh(\eta)$ in the (1+1)-dimensional Minkowski space-time. We compute the velocity :

$$u^\mu = \frac{dx^\mu}{d\tau} = \dot{\eta} \frac{\partial x^\mu}{\partial \eta} = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \end{pmatrix} \quad (1)$$

where we have used the fact that $\dot{\eta} = \rho_0^{-1}$ on the trajectory. This is the case simply because on a constant $\rho = \rho_0$ trajectory we have from the metric $d\tau^2 = \rho_0^2 d\eta^2$. Then, we check that the velocity is time-like and normalized properly computing :

$$u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -\cosh(\eta)^2 + \sinh(\eta)^2 = -1 \quad (2)$$

We compute the acceleration in the same way :

$$a^\mu = \frac{du^\mu}{d\tau} = \dot{\eta} \frac{\partial u^\mu}{\partial \eta} = \rho_0^{-1} \begin{pmatrix} \sinh(\eta) \\ \cosh(\eta) \end{pmatrix} \Rightarrow a = \sqrt{a_\mu a^\mu} = \rho_0^{-1} \quad (3)$$

and we clearly see that the norm of the acceleration is constant.

- (b) From

$$d\rho = \sqrt{\frac{2m}{r-2m}} dr \quad (4)$$

we directly have that $\rho = 2\sqrt{2m}\sqrt{r-2m}$ such that $\rho^2 = 8m(r-2m)$. Moreover, using $\eta = \frac{t}{4m}$ we get :

$$d\eta = \frac{1}{4m} dt \quad (5)$$

such that :

$$\begin{aligned} ds^2 &= -\frac{r-2m}{2m} dt^2 + \frac{2m}{r-2m} dr^2 \\ &= -16m^2 \frac{r-2m}{2m} d\eta^2 + d\rho^2 \\ &= -\rho^2 d\eta^2 + d\rho^2 \end{aligned} \quad (6)$$

for the near- r_S Schwarzschild metric.

2. KRUSKAL COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME
(SOLUTION BLAISE)

- (a) To compute the Schwarzschild metric in the new (X, T) -coordinate, it's useful to consider the two expression $t(X, T)$ and $r^*(X, T)$. To find these we first rewrite (X, T) as :

$$\begin{aligned} X &= e^{r^*/4m} \cosh(t/4m) \\ T &= e^{r^*/4m} \sinh(t/4m) . \end{aligned} \quad (7)$$

This leads in particular to :

$$X^2 - T^2 = e^{r^*/2m} = e^{r/2m} \left(\frac{r}{2m} - 1 \right) = rF(r) \frac{e^{r/2m}}{2m} \quad (8)$$

which is a way to express r implicitly (we have defined $F := \frac{\partial r}{\partial r^*} = 1 - \frac{2m}{r}$ for later use). Now, from (6) it also follows that :

$$\begin{aligned} t &= 4m \operatorname{atanh}(T/X) \\ r^* &= 2m \log(X^2 - T^2) . \end{aligned} \quad (9)$$

and this allows us to compute the partial derivative we will need :

$$\begin{aligned} \frac{\partial t}{\partial T} &= \frac{4mX}{X^2 - T^2} & \frac{\partial r}{\partial T} &= \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial T} = F \frac{4mT}{T^2 - X^2} \\ \frac{\partial t}{\partial X} &= \frac{4mT}{T^2 - X^2} & \frac{\partial r}{\partial X} &= \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial X} = F \frac{4mX}{X^2 - T^2} \end{aligned} \quad (10)$$

Then it is straightforward to compute the Schwarzschild metric starting from the old (t, r) -coordinate and we get :

$$\begin{aligned} ds^2 &= -F dt^2 + F^{-1} dr^2 + r^2 d\Omega^2 \\ &= -F \left(\frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial X} dX \right)^2 + F^{-1} \left(\frac{\partial r}{\partial T} dT + \frac{\partial r}{\partial X} dX \right)^2 + r^2 d\Omega^2 \\ &= \frac{16m^2 F}{(X^2 - T^2)^2} \left[-(X dT - T dX)^2 + (-T dT + X dX)^2 \right] + r^2 d\Omega^2 \\ &= \frac{16m^2 F}{(X^2 - T^2)} \left[-dT^2 + dX^2 \right] + r^2 d\Omega^2 \\ &= \frac{32m^3}{r} e^{-r/2m} \left[-dT^2 + dX^2 \right] + r^2 d\Omega^2 \end{aligned} \quad (11)$$

where in the last step we have used (8).

- (b) We use equation (8) which gives implicitly $r(X, T)$. Setting $r = 2m$, we see that this correspond to $X^2 = T^2$, or :

$$X = \pm T \quad (12)$$

Using the same relation as before one can find the singularity at $r = 0$ in terms of (X, T) . We find that the singularity is described by two curves :

$$X^2 - T^2 = -1 \quad (13)$$

2. KRUSKAL COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME
(SOLUTION MATTHIAS)

Write the Schwarzschild metric as

$$ds^2 = (1 - 2m/r)[-dt^2 + dr^{*2}] + r^2 d\Omega^2 = (1 - 2m/r)[-du dv] + r(u, v)^2 d\Omega^2 \quad (14)$$

where $r^* = r + 2m \log(r/2m - 1)$ is the tortoise coordinate, and $u = t + r^*$, $v = t - r^*$ are the “advanced” and “retarded” Eddington-Finkelstein coordinates. Now note that

$$\frac{u - v}{4m} = \frac{r}{2m} + \log\left(\frac{r}{2m} - 1\right) , \quad (15)$$

so that

$$1 - \frac{2m}{r} = \frac{2m}{r} \left(\frac{r}{2m} - 1\right) = \frac{2m}{r} e^{-r/2m} e^{(u-v)/4m} . \quad (16)$$

Thus the metric is

$$ds^2 = \frac{2m}{r} e^{-r/2m} \left(e^{u/4m} du\right) \left(-e^{-v/4m} dv\right) + r(u, v)^2 d\Omega^2 . \quad (17)$$

Therefore it is natural to introduce

$$U = e^{u/4m} , \quad V = -e^{-v/4m} , \quad (18)$$

and T and X via $U = T + X, V = T - X$, so that

$$\begin{aligned} ds^2 &= -\frac{32m^3}{r} e^{-r/2m} dU dV + r(u, v)^2 d\Omega^2 \\ &= \frac{32m^3}{r} e^{-r/2m} [-dT^2 + dX^2] + r(T, X)^2 d\Omega^2 . \end{aligned} \quad (19)$$

Moreover, equation (16) implies that

$$\left(\frac{r}{2m} - 1\right) e^{r/2m} = e^{(u-v)/4m} = -UV = X^2 - T^2 , \quad (20)$$

so that $r = 2m \Leftrightarrow X = \pm T$ and $r = 0 \Leftrightarrow X^2 - T^2 = -1$.

3. PAINLEVÉ-GULLSTRAND COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME

- (a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates (t, r) defining a new coordinate $T(t, r) = t + \psi(r)$. This leads us to rewrite the metric with $dT = dt + \psi' dr$ and we find :

$$ds^2 = -f(r)dT^2 + 2f(r)\psi' dT dr + f(r)^{-1}(1 - f(r)^2\psi'^2)dr^2 + r^2 d\Omega^2 \quad (21)$$

Choosing $C(r) = f(r)\psi'$ gives the desired result and the function $C(r)$ is completely arbitrary because $\psi(r)$ is arbitrary.

- (b) In the Painlevé-Gullstrand coordinate we make a particular choice for $C(r)$ namely $C(r) = \sqrt{1 - f(r)}$ such that $g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1$. We are thus left with the metric :

$$ds^2 = -f(r)dT^2 + 2\sqrt{\frac{2m}{r}}dTdr + dr^2 + r^2d\Omega^2 \quad (22)$$

Now, with this new choice of coordinate we see that any component $g_{\mu\nu}$ of the metric stays finite for any value of $r > 0$ (it was not the case at the beginning in the (t, r) -coordinate). In addition to that we also notice that the determinant of the metric is :

$$\det(g_{\mu\nu}) = \left(-f(r) - \frac{2m}{r}\right)r^4 \sin^2(\theta) = -r^4 \sin^2(\theta)^2 \quad (23)$$

which is non-vanishing for any $r > 0$ (with $\theta \neq \pm\frac{\pi}{2}$ of course).

- (c) If we now make the choice $C(r) = 1$, then the metric becomes :

$$ds^2 = -f(r)dT^2 + 2dTdr + r^2d\Omega^2 \quad (24)$$

and if we rename $T(t, r)$ by $u(t, r)$, then :

$$ds^2 = -f(r)du^2 + 2dudr + r^2d\Omega^2 \quad (25)$$

which is exactly the metric in the Eddington-Finkelstein coordinates. We can check explicitly that the transformation is indeed the same. The particular choice $C(r) = 1$ implies that $\psi(r)$ is such that $\psi' = \frac{1}{f(r)}$. But we have $\frac{\partial r^*}{\partial r} = f(r)^{-1}$, which in turn implies $\psi = r^* + cte$ where the constant can be set to zero so that we have $T(t, r) = t + \psi = t + r^* = u(t, r)$ as we should.