1. ORBIT EQUATION FOR GENERAL STATIC SPHERICALLY SYMMETRIC METRICS

   First we extract the constants of motion from the Lagrangian :

$$\begin{aligned}
\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{\epsilon}{2} &= -\frac{1}{2}A(r)\dot{t}^2 + \frac{1}{2}B(r)\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\sin(\theta)^2\dot{\phi}^2 \\
&\Rightarrow -\frac{\partial \mathcal{L}}{\partial \dot{t}} = E = A(r)\dot{t} && (1) \\
&\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L = r^2\sin(\theta)^2\dot{\phi} && (2) \\
\text{and we fix } \theta = \theta_0 = \frac{\pi}{2} &\Rightarrow \mathcal{L} = -\frac{E^2}{2A(r)} + \frac{B(r)}{2}\dot{r}^2 + \frac{L^2}{2r^2} = \frac{\epsilon}{2} && (3)
\end{aligned}$$

   We have $r = r(\tau)$ but we want to express r as a function of ϕ, as we do so this implies that $\dot{r} = r'\dot{\phi} = \frac{Lr'}{r^2}$ and we get with $r = r(\phi)$ and $r' = \frac{dr}{d\phi}$:

$$-\frac{E^2}{L^2A(r)} + B(r)\frac{r'^2}{r^4} + \frac{1}{r^2} = \frac{\epsilon}{L^2} \qquad (4)$$

   Finally, we change variable from r to $u = 1/r$ ($\Rightarrow r' = -\frac{u'}{u^2}$) and (4) becomes :

$$\tilde{B}(u)u'^2 + u^2 = \frac{\epsilon}{L^2} + \frac{E^2}{L^2\tilde{A}(u)} \qquad (5)$$

   where $\tilde{B}(u) = B(r(u)) = B(\frac{1}{u})$ and the same for $\tilde{A}(u)$.

2. RADIAL FALL & THE REPULSIVE REISSNER-NORDSTRØM CORE

   We use equation (3) with $\epsilon = -1$, $A = B^{-1} = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$ and $L = 0$ which after multiplying both sides by A gives :

$$\begin{aligned}
-\frac{E^2}{2} + AB\frac{\dot{r}^2}{2} &= -\frac{A}{2} && (6) \\
\frac{\dot{r}^2}{2} &= \frac{E^2}{2} - \frac{A}{2} && (7) \\
\frac{\dot{r}^2}{2} - \frac{m}{r} + \frac{q^2}{2r^2} &= \frac{E^2}{2} - \frac{1}{2} && (8) \\
\frac{\dot{r}^2}{2} + V_{eff}(r) &= \mathcal{E} && (9)
\end{aligned}$$

   We observe that for $q \neq 0$ the dominant term in the effective potential for a small radius is now positive and we have $V_{eff}(r \to 0) \to \infty$ such that infalling particles cannot reach $r = 0$ (as they would need an infinite amount of energy).

3. Tensor Analysis IV: The Riemann Curvature Tensor

(a) On a scalar we have $\nabla_\mu \phi = \partial_\mu \phi$, therefore :

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\,\phi &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\phi && (10)\\ &= (\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu)\,\phi && (11)\\ &= (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\,\phi - \left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right)\partial_\rho \phi && (12)\\ &= 0 && (13) \end{aligned}$$

(b) We just have to compute :

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] V^\lambda &= \nabla_\mu \left[\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\rho} V^\rho\right] - (\ \mu \leftrightarrow \nu\ ) \\ &= \partial_\mu \partial_\nu V^\lambda + \partial_\mu \Gamma^\lambda_{\nu\rho} V^\rho + \Gamma^\lambda_{\mu\rho} \partial_\nu V^\rho - \Gamma^\rho_{\mu\nu} \partial_\rho V^\lambda + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - \Gamma^\sigma_{\mu\nu} \Gamma^\lambda_{\sigma\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\ &= \partial_\mu \Gamma^\lambda_{\nu\rho} V^\rho + \Gamma^\lambda_{\mu\rho} \partial_\nu V^\rho + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\ &= (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \Gamma^\lambda_{\nu\rho} \partial_\mu V^\rho + \Gamma^\lambda_{\mu\rho} \partial_\nu V^\rho + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\ &= (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\ \mu \leftrightarrow \nu\ ) \\ &= \left[\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho}\right] V^\rho = R^\lambda{}_{\rho\mu\nu} V^\rho && (14) \end{aligned}$$

where in the third equality we dropped all the (μ, ν)-symmetric terms killed by the subtraction with the indices exchanged.