Solutions to Assignments 04 — 10.11.09

1. Stationary observers and lightrays in the Schwarzschild geometry

   (a) The observer is sitting at fixed radius and angles, therefore his worldline 4-velocity is of the form :

$$\frac{dx^\mu}{d\tau} = u^\mu = \begin{pmatrix} u^t \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ with: } u_\mu u^\mu = -1, \text{ and } u^t > 0 \Rightarrow u^t = \frac{1}{\sqrt{1-\frac{2m}{r}}} \quad (1)$$

The acceleration is then :

$$\begin{aligned} a^\mu = \nabla_\tau u^\mu &= u^\rho \nabla_\rho u^\mu \\ &= u^t \partial_t u^\mu + u^t \Gamma^\mu_{tt} u^t \\ &= \Gamma^\mu_{tt} \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} g^{\mu\rho} \partial_\rho g_{tt} \frac{1}{1-\frac{2m}{r}} \\ \text{for } \mu \neq r &= 0 \\ \text{for } \mu = r &= \frac{1}{2} g^{rr} \partial_r (1 - \frac{2m}{r}) \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} \partial_r \frac{2m}{r} = \frac{m}{r^2} \end{aligned} \quad (2)$$

and finally the norm of the acceleration is :

$$\begin{aligned} g_{\mu\nu} a^\mu a^\nu &= g_{rr} a^r a^r \\ &= \frac{1}{1-\frac{2m}{r}} \frac{m^2}{r^4} \end{aligned} \quad (3)$$

   (b) For lightrays we have $ds^2 = 0$, and because they are radial we get :

$$\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1-\frac{2m}{r}} dr^2 = 0 \;\Rightarrow\; dt = \pm \frac{1}{1-\frac{2m}{r}} dr \quad (4)$$

which can be integrated to give the solution ($c_\pm$ are constants of integration)

$$t_\pm(r) = \pm(r + 2m \log(r - 2m)) + c_\pm \quad (5)$$

2. Tensor Analysis III: The Covariant Divergence

(a) First we compute $\Gamma^{\mu}_{\mu\lambda}$ with the definition :

$$\begin{aligned}\Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) \\ &= \frac{1}{2}(\partial^\rho g_{\rho\lambda} + g^{\mu\rho}\partial_\lambda g_{\mu\rho} - \partial^\mu g_{\mu\lambda}) \\ &= \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\mu\rho} \end{aligned} \tag{6}$$

Then, we use the relation $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find that :

$$\begin{aligned}\Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\mu\rho} \\ &= \frac{1}{2}g^{-1}\partial_\lambda g \\ &= g^{-1/2}\partial_\lambda g^{+1/2} \end{aligned} \tag{7}$$

where in the last equality we used the fact that : $\partial_\lambda g^{+1/2} = \frac{1}{2}g^{-1/2}\partial_\lambda g$.

(b) We can now compute the covariant divergence :

$$\begin{aligned}\nabla_\mu J^\mu &= \partial_\mu J^\mu + \Gamma^{\mu}_{\mu\rho}J^\rho \\ &= \partial_\mu J^\mu + J^\rho g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \end{aligned} \tag{8}$$

and

$$\begin{aligned}\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}F^{\rho\nu} + \Gamma^{\nu}_{\mu\rho}F^{\mu\rho} \\ &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu}g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \end{aligned} \tag{9}$$

where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^{\nu}_{\mu\rho}$). More precisely, if we rewrite $\Gamma^{\nu}_{\mu\rho} = \frac{1}{2}(\Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\rho\mu})$ and $F^{\mu\rho} = \frac{1}{2}(F^{\mu\rho} - F^{\rho\mu})$, then $\Gamma^{\nu}_{\mu\rho}F^{\mu\rho}$ contains 4 terms and relabelling two of them by the exchange of the indices $\mu \leftrightarrow \rho$ we see that everything vanishes.