Solutions to Assignments 03

03.11.09

Tensor Analysis II: The Covariant Derivative

1. Consider the scalar $A_\nu V^\nu$ and take its covariant derivative :

$$\begin{aligned} \nabla_\mu (A_\nu V^\nu) &= \partial_\mu (A_\nu V^\nu) = A_\nu \partial_\mu V^\nu + V^\nu \partial_\mu A_\nu & (1) \\ &= A_\nu \nabla_\mu V^\nu + V^\nu \nabla_\mu A_\nu & (2) \end{aligned}$$

This implies :

$$\begin{aligned} V^\nu \nabla_\mu A_\nu &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu \nabla_\mu V^\nu & (3) \\ &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu (\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) & (4) \\ &= V^\nu \partial_\mu A_\nu - A_\nu \Gamma^\nu_{\mu\rho} V^\rho = V^\nu \partial_\mu A_\nu - A_\lambda \Gamma^\lambda_{\mu\nu} V^\nu & (5) \\ &\Rightarrow \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda & (6) \end{aligned}$$

2. Using the same notation as in the assignments 02, we see that $\partial_\mu A_\nu$ isn't a tensor because it transforms like :

$$\begin{aligned} \partial_\mu A_\nu \to \partial_{\mu'} A_{\nu'} &= J^\mu_{\mu'} \partial_\mu (J^\nu_{\nu'} A_\nu) & (7) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\mu_{\mu'} \partial_\mu J^\nu_{\nu'} & (8) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\nu_{\mu'\nu'} & (9) \\ &\neq J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu & (10) \end{aligned}$$

where I have defined :

$$J^\nu_{\mu'\nu'} = J^\mu_{\mu'} \partial_\mu J^\nu_{\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^{\nu'}} = \frac{\partial^2 x^\nu}{\partial y^{\mu'} \partial y^{\nu'}} \qquad (11)$$

Now take the combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we see that $F_{\mu\nu}$ transforms like a tensor (and this is due to the fact that $J^\nu_{\mu'\nu'}$ is symmetric under the permutation $\mu' \leftrightarrow \nu'$ whereas $F_{\mu'\nu'}$ is anti-symmetric) :

$$\begin{aligned} F_{\mu\nu} \to \partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\nu_{\mu'\nu'} - J^\nu_{\nu'} J^\mu_{\mu'} \partial_\nu A_\mu - A_\mu J^\mu_{\nu'\mu'} & (12) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} (\partial_\mu A_\nu - \partial_\nu A_\mu) + A_\nu (J^\nu_{\mu'\nu'} - J^\nu_{\nu'\mu'}) & (13) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} F_{\mu\nu} & (14) \end{aligned}$$

The identity $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ follows directly from the symmetry of the Christofell symbols for the two bottom indices $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ so that they cancel out in the $F_{\mu\nu}$ term :

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \qquad (15)$$

3. See lecture notes equation (4.30).