SOLUTIONS TO ASSIGNMENTS 02

27.10.09

1. GEODESICS

(a) Take the Lagrangian to be :

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{1}$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ and then compute :

$$\begin{align} \frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{1}{2} \dot{x}^\rho \dot{x}^\nu \partial_\mu g_{\rho\nu} \tag{2} \\ \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} &= g_{\rho\nu} \dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu} \dot{x}^\nu = g_{\rho\nu} \dot{x}^\rho \delta^\nu_\mu = g_{\rho\mu} \dot{x}^\rho \tag{3} \\ \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) &= g_{\rho\mu} \ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} = g_{\rho\mu} \ddot{x}^\rho + \frac{1}{2} \left(\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\rho g_{\nu\mu} \right) \tag{4} \end{align}$$

The Euler-Lagrange equations becomes :

$$\begin{align} \Big[\text{E.-L.} \Big] &= g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} \left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu} \right) \dot{x}^\nu \dot{x}^\rho \tag{5} \\ &= g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0 \tag{6} \end{align}$$

and they can be written in the usual (geodesic equations) form by multiplying by $g^{\lambda\mu}$ to move the index μ up :

$$g^{\lambda\mu} \left(g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\rho \dot{x}^\nu \right) = \ddot{x}^\lambda + \Gamma^\lambda_{\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0 \tag{7}$$

(b) First we compute :

$$\frac{d}{d\tau} \mathcal{L} = \frac{1}{2} \frac{d}{d\tau} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \Big(2 g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu + \, (\dot{x}^\rho \partial_\rho g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu \Big) \tag{8}$$

Now using the identity :

$$\partial_\rho g_{\mu\nu} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} \tag{9}$$

together with the fact that $x^\mu(\tau)$ is a solution to the geodesic equation, which means that we also have :

$$\ddot{x^\mu} = -\Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho \tag{10}$$

leaves us with :

$$\begin{align} \frac{d}{d\tau} \mathcal{L} &= \frac{1}{2} \Big(-2 g_{\mu\nu} \Gamma^\mu_{\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + \dot{x}^\rho (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu \Big) \tag{11} \\ &= \frac{1}{2} \Big(-2 \Gamma_{\nu\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu \dot{x}^\rho \Big) = 0 \tag{12} \end{align}$$

witch is obviously zero if we relabel the indices.

(c) The metric on the 2-sphere is :

$$ds^2 = R^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \tag{13}$$

so that in the (θ, ϕ) coordinates we have :

$$g_{\mu\nu} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \qquad g^{\mu\nu} = R^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^{-2} \end{pmatrix} \tag{14}$$

Because $g_{\mu\nu}$ is diagonal and $g_{\mu\nu} = g_{\mu\nu}(\theta)$, the only non-vanishing contribution to the Christoffel's will come from $\partial_\theta g_{\phi\phi}$. Keeping this in mind, we can find out all the Christoffel symbols that have this term. We first compute terms with θ on the top :

$$\Gamma^{\theta}_{\nu\lambda} = \frac{1}{2} g^{\theta\theta} \left(\partial_\lambda g_{\theta\nu} + \partial_\nu g_{\theta\lambda} - \partial_\theta g_{\nu\lambda} \right) = -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\nu\lambda} \tag{15}$$

and we see that the only non-vanishing term with θ on the top is $\Gamma^{\theta}_{\phi\phi}$:

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\phi\phi} = -\sin(\theta)\cos(\theta) \tag{16}$$

Now if we choose ϕ to be on the top :

$$\Gamma^{\phi}_{\nu\lambda} = \frac{1}{2} g^{\phi\phi} \left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda} - \partial_\phi g_{\nu\lambda} \right) = \frac{1}{2} g^{\phi\phi} \left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda} \right) \tag{17}$$

The only non-vanishing terms with ϕ on the top are $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi}$:

$$\Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{18}$$

We use the computed Christoffel's to write down the geodesic equation $\ddot{x}^\mu + \Gamma^{\mu}_{\nu\rho} \dot{x}^\nu \dot{x}^\rho$ in the (θ, ϕ) coordinate. We find :

$$\begin{cases} \ddot{\theta} + \Gamma^{\theta}_{\phi\phi} \dot{\phi}\dot{\phi} = \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}\dot{\phi} = 0 & \mu = \theta \\ \\ \ddot{\phi} + 2\Gamma^{\phi}_{\theta\phi} \dot{\theta}\dot{\phi} = \ddot{\phi} + 2\dfrac{\cos(\theta)}{\sin(\theta)} \dot{\theta}\dot{\phi} = 0 & \mu = \phi \end{cases} \tag{19}$$

Using the Euler-Lagrange equations with $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \sin(\theta)^2 \dot{\phi}^2)$, we get :

$$\begin{cases} \dfrac{d}{d\tau} \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dfrac{\partial \mathcal{L}}{\partial \theta} = \dfrac{1}{2} \left(\dfrac{d}{d\tau} 2\dot{\theta} - 2\sin(\theta)\cos(\theta)\dot{\phi}^2 \right) = 0 \\ \\ \dfrac{d}{d\tau} \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} - \dfrac{\partial \mathcal{L}}{\partial \phi} = \dfrac{1}{2} \dfrac{d}{d\tau} \left(2\sin(\theta)^2 \dot{\phi} \right) = \sin(\theta)^2 \ddot{\phi} + 2\cos(\theta)\sin(\theta)\dot{\theta}\dot{\phi} = 0 \end{cases} \tag{20}$$

which are the same equations as in (19).

We can easily see that the great circles $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ on S^2 are solutions to the equations just found. It is indeed the case because for that particular solution ϕ is constant along a great circle, witch implies $\ddot{\phi} = \dot{\phi} = 0$ and simultaneously $\theta(\tau) = \tau$ so that $\ddot{\theta}$ vanishes. By looking at equation (19) or (20) we can see that every curb with $\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0$ trivially satisfy both equations and therefore curbs like that are geodesics.

2. Tensor analysis I: Tensor Algebra

(a) Under a coordinate transformation $x^\mu \to y^{\mu'}(x^\mu)$ a function transforms such that his value at a fixed point remains the same : $f(x^\mu) \to f'(y^{\mu'}) = f(x^\mu)$. For the partial derivative of a function we have

$$\frac{\partial}{\partial x^\mu} f(x^\mu) \to \frac{\partial}{\partial y^{\mu'}} f'(y^{\mu'}) = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial}{\partial x^\mu} f(x^\mu) = J^\mu_{\mu'} \frac{\partial}{\partial x^\mu} f(x^\mu) \tag{21}$$

which shows that $\partial_\mu f$ is a covector. To simplify notation we define

$$J^\mu_{\mu'} = \frac{\partial x^\mu}{\partial y^{\mu'}} \qquad \text{and} \qquad J^{\mu'}_\mu = \frac{\partial y^{\mu'}}{\partial x^\mu} \tag{22}$$

which are the component of the Jacobian of the coordinate transformation and it's inverse.

(b) By definition ($A_{\mu\nu}$ and B^μ being tensors) we have :

$$A_{\mu\nu}B^\nu \to A_{\mu'\nu'}B^{\nu'} = \left(J^\mu_{\mu'} J^\nu_{\nu'} A_{\mu\nu}\right)\left(J^{\nu'}_\rho B^\rho\right) = J^\mu_{\mu'} \delta^\nu_\rho A_{\mu\nu} B^\rho = J^\mu_{\mu'} \left(A_{\mu\nu} B^\nu\right) \tag{23}$$

so that $A_{\mu\nu}B^\nu$ transforms like a co-vector. The same goes for $A_{\mu\nu}B^\mu B^\nu$:

$$A_{\mu\nu}B^\mu B^\nu \to \left(J^\mu_{\mu'} J^\nu_{\nu'} A_{\mu\nu}\right)\left(J^{\mu'}_\rho B^\rho\right)\left(J^{\nu'}_\lambda B^\lambda\right) = \delta^\mu_\rho \delta^\nu_\lambda A_{\mu\nu} B^\rho B^\lambda = A_{\mu\nu} B^\mu B^\nu \tag{24}$$

witch transforms like a scalar.

(c) The invariance of $V(x)$ under coordinate transformations follows from the fact that partial derivatives are covectors and that they are contracted with a vector to form the field $V(x)$. By the chain rule, we see (again) that :

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \to \partial_{\mu'} = \frac{\partial}{\partial y^{\mu'}} = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial}{\partial x^\mu} = J^\mu_{\mu'} \partial_\mu \tag{25}$$

and with this we have :

$$V(x) \to V'(x) = V^{\mu'}(x)\, \partial_{\mu'} = (J^{\mu'}_\mu V^\mu)(J^\nu_{\mu'} \partial_\nu) = \delta^\nu_\mu V^\mu \partial_\nu = V(x) \tag{26}$$

To see that $A(x)$ is invariant too, we have to show that dx^μ is a vector :

$$dx^\mu \to dy^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\mu} dx^\mu \tag{27}$$

and thus :

$$A(x) = A_\mu(x) dx^\mu \to A_{\mu'} dy^{\mu'} = (J^\nu_{\mu'} A_\nu)(J^{\mu'}_\mu dx^\mu) = A(x) \tag{28}$$

is also invariant under coordinate transformations.