

GR ASSIGNMENTS 03

0. GEODESIC LAGRANGIAN: EXAMPLES AND APPLICATIONS

As described in the Lecture Notes (Section 3.1), the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \quad (1)$$

is the ideal starting point to

- determine the geodesic equations
- determine all the Christoffel symbols
- read off some conserved charges (first integrals) from cyclic coordinates
- in some cases read off some geodesics by inspection of the Lagrangian

To practice this, do this for the following physically important metrics:

- (a) the Rindler metric

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \quad (2)$$

(what is the symmetry of Minkowski space corresponding to invariance under η -translations?)

- (b) the general static spherically symmetric in the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3)$$

(we will need to know all the Christoffel symbols later on, when deriving the Schwarzschild metric as a solution of the Einstein equations, so you may as well determine them now ...)

- (c) the spatially flat cosmological (Friedmann-Robertson-Walker) metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \quad (4)$$

(again we will need all the Christoffel symbols later on; also learn how to see without any calculation that the lines of constant $\vec{x}$ are geodesics - those of so-called “comoving observers”)

(Solutions will not be provided, but can for the most part be found scattered throughout the notes if you really need to look them up.)

1. TENSOR ALGEBRA

Let $f(x)$ be a scalar field, $V^\alpha(x)$ a vector field, $A_\alpha(x)$ a covector field, and denote by $\partial_\alpha = \partial/\partial x^\alpha$ the partial derivatives.

- (a) Show that $\partial_\alpha f$ is (i.e. transforms like) a covector field.

(b) Show that the first-order linear differential operator

$$V(x) = V^\alpha(x) \partial_\alpha \quad (5)$$

is *invariant* under coordinate transformations.

(c) Analogously, show that

$$A(x) = A_\alpha(x) dx^\alpha \quad (6)$$

is invariant under coordinate transformations.

Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V . V can be expanded in a basis ∂_α , and its components with respect to this basis are the V^α . If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.

In order to understand in a different way why vector fields are naturally to be thought of as differential operators $V = V^\alpha \partial_\alpha$, note that the integral curves $x^\alpha = x^\alpha(\lambda)$ of a vector field V are the solutions to the differential equation

$$\frac{d}{d\lambda} x^\alpha(\lambda) = V^\alpha(x(\lambda)) \quad (7)$$

Then the change of any function (scalar) $f(x)$ along the flow (integral curves) of the vector field is given by the action of the differential operator $V = V^\alpha \partial_\alpha$ on f ,

$$\frac{d}{d\lambda} f(x(\lambda)) = \frac{\partial f}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} = V^\alpha \partial_\alpha f(x) \quad (8)$$

Likewise, the above argument shows that covector fields are most naturally (and invariantly) to be thought of as objects that can (and want to) be integrated over one-dimensional curves, with $\oint A(x) = \oint A_\alpha dx^\alpha$ independent of the choice of coordinates.

2. THE EFFECTIVE GEODESIC POTENTIAL

Generalising the discussion in section 25.3 of the lecture notes, and following Remark 6 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \quad (9)$$

(this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).