

# GR ASSIGNMENTS 04

## 1. TENSOR ANALYSIS I: THE COVARIANT DERIVATIVE

The covariant derivatives of a covector field  $A_\mu$  and a  $(0, 2)$  tensor  $B_{\mu\nu}$  field are

$$\begin{aligned}\nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma_{\mu\nu}^\lambda A_\lambda \\ \nabla_\lambda B_{\mu\nu} &= \partial_\lambda B_{\mu\nu} - \Gamma_{\lambda\mu}^\rho B_{\rho\nu} - \Gamma_{\lambda\nu}^\rho B_{\mu\rho} .\end{aligned}\tag{1}$$

- (a) Show that, even though  $\partial_\mu A_\nu$  is *not* a tensor (which is why the  $\Gamma$ -term is required in  $\nabla_\mu A_\nu$ ), the *curl* (or *rotation*)  $\partial_\mu A_\nu - \partial_\nu A_\mu$  is (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu .\tag{2}$$

This provides an alternative argument for the fact that  $\partial_\mu A_\nu - \partial_\nu A_\mu$  is a tensor.

- (b) Show that the covariant derivative of the metric is zero,  $\nabla_\lambda g_{\mu\nu} = 0$ .

## 2. STATIONARY AND FREELY FALLING SCHWARZSCHILD OBSERVERS

- (a) Consider a stationary observer (sitting at fixed values of  $(r > 2m, \theta, \phi)$ ) in the Schwarzschild geometry

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) .\tag{3}$$

Determine her worldline 4-velocity  $u^\alpha = dx^\alpha/d\tau$  and the covariant (tensorial, specifically vectorial) acceleration  $a^\alpha = \dot{u}^\alpha + \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma$  and calculate  $g_{\alpha\beta} a^\alpha a^\beta$ .

- (b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius  $r(\tau = 0) \equiv R > 2m$ . Show that the proper time it would (formally) take her to reach  $r = 0$  is (up to factors of  $c$ ) given by

$$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} .\tag{4}$$

Estimate this for  $R$  the radius of the sun ( $R \sim 7 \times 10^{10}$  cm) and  $2m$  its Schwarzschild radius ( $2m \sim 3 \times 10^5$  cm), restoring the correct factors of  $c$ , and show that this is of the order of an hour.

**Remark:** this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.