GR Assignments 03

1. Tensor Algebra

   Let $f(x)$ be a scalar field, $V^\alpha(x)$ a vector field, $A_\alpha(x)$ a covector field, and denote by $\partial_\alpha = \partial/\partial x^\alpha$ the partial derivatives.

   (a) Show that $\partial_\alpha f$ is (i.e. transforms like) a covector field.

   (b) Show that the first-order linear differential operator

   $$V(x) = V^\alpha(x)\,\partial_\alpha \tag{1}$$

   is *invariant* under coordinate transformations.

   (c) Analogously, show that

   $$A(x) = A_\alpha(x)dx^\alpha \tag{2}$$

   is invariant under coordinate transformations.

   Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V. V can be expanded in a basis ∂_α, and its components with respect to this basis are the V^α. If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.

   In order to understand in a different way why vector fields are naturally to be thought of as differential operators $V = V^\alpha\partial_\alpha$, note that the integral curves $x^\alpha = x^\alpha(\lambda)$ of a vector field V are the solutions to the differential equation

   $$\frac{d}{d\lambda}x^\alpha(\lambda) = V^\alpha(x(\lambda)) \ . \tag{3}$$

   Then the change of any function (scalar) $f(x)$ along the flow (integral curves) of the vector field is given by the action of the differential operator $V = V^\alpha\partial_\alpha$ on f,

   $$\frac{d}{d\lambda}f(x(\lambda)) = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = V^\alpha\partial_\alpha f(x) \tag{4}$$

   Likewise, the above argument shows that covector fields are most naturally (and invariantly) to be thought of as objects that can (and want to) be integrated over one-dimensional curves, with $\oint A(x) = \oint A_\alpha dx^\alpha$ independent of the choice of coordinates.

2. The Effective Geodesic Potential

   Generalising the discussion in section 25.3 of the lecture notes, and following Remark 6 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{5}$$

   (this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).