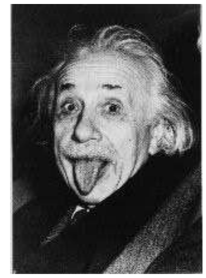

GR Assignments 01

1. Metrics, Line Elements and Coordinate Transformations

   Under a cooordinate transformation from Cartesian (or inertial) coordinates $\xi^a = \xi^a(x^\mu)$, the Euclidean metric δ_{ab} or the Minkowski metric η_{ab} transforms to a new metric $g_{\mu\nu}$ in such a way that proper distances are invariant,

$$(\delta_{ab} \text{ or } \eta_{ab}) d\xi^a d\xi^b = g_{\mu\nu} dx^\mu dx^\nu \quad \Leftrightarrow \quad g_{\mu\nu} = J^a_\mu J^b_\nu (\delta_{ab} \text{ or } \eta_{ab}) \;, \tag{1}$$

   with $J^a_\mu = \partial\xi^a/\partial x^\mu$ the Jacobi matrix of the transformation.

   (a) 2-dimensional Metrics

   Consider the 2-dimensional line element

$$ds^2 = (dy^1)^2 + (dy^2)^2 + 2a dy^1 dy^2 \;, \tag{2}$$

   with $a \in \mathbb{R}$ a real constant parameter.

   - Show that this metric is non-degenerate for $a \neq \pm 1$.
   - Show that for $a^2 < 1$ the metric is related to the standard Euclidean metric $ds^2 = (dx^1)^2 + (dx^2)^2$ by the coordinate transformation

$$x^1 = \sqrt{1-a^2}\, y^1 \quad , \quad x^2 = a y^1 + y^2 \;. \tag{3}$$

   - Show that for $a^2 > 1$ the metric is related to the standard Lorentzian metric $ds^2 = -(dt)^2 + (dx)^2$ by a coordinate transformation.

   (b) Rindler Metric

   Consider the (1+1)-dimensional Minkowski metric with line element $ds^2 = -dt^2 + dx^2$, and the *Rindler coordinates* (T, X), defined by

$$t = X \sinh T \quad , \quad x = X \cosh T \;. \tag{4}$$

   - Determine the metric or line element in these coordinates.
   - Show that the lines of constant T or constant X are straight lines through the origin or hyperbolae (in a (t, x)-diagram) respectively.

2. GEODESICS I

(a) GEODESICS AND EULER-LAGRANGE EQUATIONS:

Show that the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial x'^{\mu}} - \frac{\partial \mathcal{L}}{\partial x^{\mu}} = 0 \ , \tag{5}$$

for the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \equiv \tfrac{1}{2} g_{\mu\nu} x'^{\mu} x'^{\nu} \tag{6}$$

are the geodesic equations

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\ \nu\lambda}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\lambda}}{d\lambda} = 0 \ , \tag{7}$$

where the *Christoffel symbols* $\Gamma^{\mu}_{\ \nu\lambda}$ associated to a metric $g_{\mu\nu}$ are defined by

$$\Gamma^{\mu}_{\ \nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda} = \tfrac{1}{2} g^{\mu\rho}(g_{\rho\nu},_{\lambda} + g_{\rho\lambda},_{\nu} - g_{\nu\lambda},_{\rho}) \tag{8}$$

(b) $\mathcal{L}$ IS A CONSTANT OF MOTION:

Show that $\mathcal{L}$ is constant along any geodesic, i.e. that

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right) = 0 \tag{9}$$

for $x^{\mu}(\lambda)$ a solution of the geodesic equation.

[As an aside: what is the symmetry that, by Noether's theorem, gives rise to this constant of motion?]