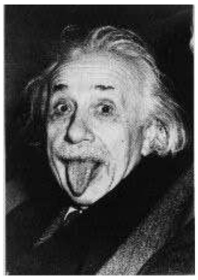

GR Assignments 01

1. Free Relativistic Particle in Arbitrary Coordinates

   As you saw in the lecture, transforming from inertial (Minkowski) coordinates ξ^a to arbitrary coordinates x^μ, the free-particle equation of motion $d^2\xi^a/d\tau^2 = 0$ becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{1}$$

   where

$$\Gamma^\mu_{\nu\lambda} = \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial x^\nu \partial x^\lambda} \equiv J^\mu_a J^a_{\nu\lambda} \ . \tag{2}$$

   Show that this pseudo-force term Γ arising in the x^μ coordinate system is related to the metric

$$g_{\mu\nu} = J^a_\mu J^b_\nu \eta_{ab} \tag{3}$$

   in these coordinates by

$$\begin{aligned} \Gamma^\mu_{\nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} \\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_\lambda + g_{\mu\lambda},_\nu - g_{\nu\lambda},_\mu) \end{aligned} \tag{4}$$

   where $g^{\mu\nu}$ is the inverse metric and $g_{\mu\nu},_\lambda$ is short-hand for the partial derivative $(\partial g_{\mu\nu}/\partial x^\lambda)$.

2. Geodesics

   (a) Geodesics and Euler-Lagrange Equations: Show that the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \ , \tag{5}$$

   for the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{6}$$

   are the geodesic equations

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ , \tag{7}$$

   where the *Christoffel symbols* $\Gamma^\mu_{\nu\lambda}$ associated to a metric $g_{\mu\nu}$ are defined as in Exercise 1 (eq. 4), but now for an arbitrary metric.

(b) $\mathcal{L}$ IS A CONSTANT OF MOTION: Show that $\mathcal{L}$ is constant along any geodesic, i.e. that

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0 \tag{8}$$

for $x^\mu(\tau)$ a solution of the geodesic equation.

(c) GEODESICS ON THE TWO-SPHERE S^2: The metric of a 2-sphere with radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{9}$$

Calculate all its Christoffel symbols, show that the geodesic equations agree with the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ , \tag{10}$$

and show that the great circles (longitudes) $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ satisfy the geodesic equation.