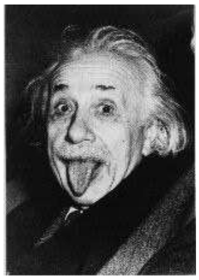

GR Assignments 05

1. On the Klein-Gordon Field in a Curved Space-Time

   The action of a real (free, massice) scalar field ϕ in a gravitational background $g_{\alpha\beta}$ is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \tag{1}$$

   The corresponding generally covariant energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} L \tag{2}$$

   (a) Derive the equation of motion

$$\left(\Box - m^2 \right) \phi = 0 \tag{3}$$

   ($\Box = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$) for ϕ from variation of the action (1) with respect to ϕ.

   (b) Show that $T_{\alpha\beta}$ is conserved when ϕ is a solution to the Klein-Gordon equation of motion,

$$\left(\Box - m^2 \right) \phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{4}$$

   (c) Show that $T_{\alpha\beta}$ is related to the variation of the action *with respect to the metric* by

$$\delta S = -\tfrac{1}{2} \int \sqrt{g} d^4x \; T_{\alpha\beta} \delta g^{\alpha\beta} \tag{5}$$

   Hint: use the variational formula $\delta\sqrt{g} = \sqrt{g} g^{\alpha\beta} \delta g_{\alpha\beta}/2$ as well as the (hopefully evident) identity $g^{\alpha\beta} \delta g_{\alpha\beta} = -g_{\alpha\beta} \delta g^{\alpha\beta}$.

2. On the Maxwell Equations in Curved Space-Time

   The Maxwell action in a gravitational background is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \tag{6}$$

   The gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta^{\ \gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \tag{7}$$

(a) Derive the vacuum Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0 \tag{8}$$

by variation of the action with respect to the gauge field A_μ.

Remark: You can write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ either in terms of partial or in terms of covariant derivatives. In the former case, you will need to use the equation $\nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu})$ for the covariant divergence in terms of $\sqrt{g}$. In the latter case, you can use the fact that $\sqrt{g}$ is covariantly constant ($\nabla_\mu\sqrt{g} = 0$).

(b) Use the vacuum Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0 \ , \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0 \ \Leftrightarrow \ \nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0 \tag{9}$$

to deduce the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{10}$$

Remark: This is a tensorial equation. For such calculations you should just use the properties of the covariant derivative and **not** write out the covariant derivative in terms of the non-tensorial Christoffel symbols and partial derivatives.

Hint: Instead of embarking blindly on this calculation, remind yourself first how to do the calculation in Minkowski space. Exactly the same procedure should then work in general. If done correctly, this should be a four-line calculation.

(c) Show that the energy momentum tensor is related to the variation of the Maxwell action with respect to the metric in the same way as for the scalar field in (5).

Hint: don't forget the implicit metric-dependence in expressions like $F_{\alpha\beta}F^{\alpha\beta}$.