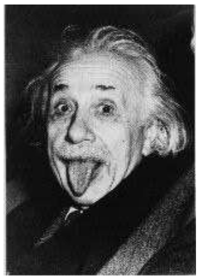

GR Assignments 04

1. Tensor Analysis II: the Covariant Derivative

The *covariant derivative* ∇_μ is the tensorial generalisation of the partial derivative ∂_μ, i.e. it is such that the covariant derivative of a tensor is again a tensor. More precisely, the covariant derivative of a (p,q)-tensor is then a $(p,q+1)$-tensor because it has one more lower (covariant) index. Since the partial derivative $\partial_\mu f$ of a scalar f (a $(0,0)$-tensor) is a covector (a $(0,1)$-tensor), one sets $\nabla_\mu f = \partial_\mu f$. However, as we have seen, the partial derivative $\partial_\mu V^\nu$ of a vector V^ν (a $(1,0)$-tensor) is *not* a $(1,1)$-tensor. This can be rectified by defining the covariant derivative of a vector to be

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad . \tag{1}$$

It can be checked that this is indeed a tensor, the non-tensorial nature of the partial derivative of a vector cancelling exactly against that of the Christoffel symbols. A smilar story holds for covectors: the partial derivative $\partial_\mu A_\nu$ of a $(0,1)$-tensor (covector) is *not* a tensor. This can be cured in the same way as for vectors, and one can check that

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{2}$$

is indeed a $(0,2)$-tensor. The action of ∇_μ on vectors and covectors can be extended to arbitrary (p,q)-tensors. For instance, for a $(0,2)$-tensor $B_{\mu\nu}$ one has

$$\nabla_\lambda B_{\mu\nu} = \partial_\lambda B_{\mu\nu} - \Gamma^\rho_{\ \lambda\mu} B_{\rho\nu} - \Gamma^\rho_{\ \lambda\nu} B_{\mu\rho} \quad . \tag{3}$$

(a) An alternative way to arrive at (2) is to demand the *Leibniz rule* for the covariant derivative of a product of tensors: deduce (2) from (1) using the fact that $A_\nu V^\nu$ is a scalar for any vector V^ν, so that $\nabla_\mu(A_\nu V^\nu) = \partial_\mu(A_\nu V^\nu)$ and using the Leibniz rule for ∂_μ (i.e. $\partial_\mu(A_\nu V^\nu) = (\partial_\mu A_\nu)V^\nu + A_\nu \partial_\mu V^\nu$) and ∇_μ.

(b) Check that, even though $\partial_\mu A_\nu$ is *not* a tensor, the *curl* (or *rotation*) $\partial_\mu A_\nu - \partial_\nu A_\mu$ *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{4}$$

This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.

(c) Show that (3) implies that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

2. Stationary and Freely Falling Schwarzschild Observers

(a) Consider a stationary observer (sitting at fixed values of $(r > 2m, \theta, \phi)$) in the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \qquad (5)$$

Determine his worldline 4-velocity $u^\alpha = dx^\alpha/d\tau$ and the acceleration $a^\alpha = \nabla_\tau u^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ and calculate $g_{\alpha\beta} a^\alpha a^\beta$. What happens as $r \to \infty$ and $r \to 2m$?

(b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of c) given by

$$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} \quad . \qquad (6)$$

Estimate this for R the radius of the sun ($R \sim 7 \times 10^{10}$ cm) and $2m$ its Schwarzschild radius ($2m \sim 3 \times 10^5$ cm), restoring the correct factors of c, and show that this is of the order of an hour.

Remark: this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.