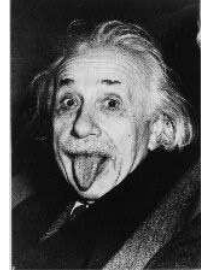


## GR ASSIGNMENTS 05



### 1. TENSOR ANALYSIS III: THE COVARIANT DIVERGENCE AND THE LAPLACIAN

An extremely useful identity for the variation (in particular, the derivative) of the determinant  $g := |\det g_{\mu\nu}|$  of the metric is

$$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \quad g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} . \quad (1)$$

One proof of this result, based on the explicit expansion of a determinant in terms of minors,  $g = \sum_\nu (-1)^{\mu+\nu} g_{\mu\nu} |m_{\mu\nu}|$  is described in the lecture notes (section 4.6). Another easy proof can be deduced from the fundamental matrix identity  $\det M = \exp \operatorname{tr} \log M$  (see the supplement to these exercises).

- (a) Use (1) to show that the contracted Christoffel symbol  $\Gamma^\mu_{\mu\lambda}$  (summation over the index  $\mu$ ) can be calculated by the simple formula

$$\Gamma^\mu_{\mu\lambda} = g^{-1/2}\partial_\lambda g^{+1/2} . \quad (2)$$

- (b) Show that this implies that the covariant divergence of a vector (current)  $J^\mu$  and an anti-symmetric tensor  $F^{\mu\nu} = -F^{\nu\mu}$  can be written as

$$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \quad \nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) . \quad (3)$$

- (c) Use the formula

$$\square\Phi = g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu\Phi) \quad (4)$$

to determine the Laplacian in  $\mathbb{R}^3$  in spherical coordinates  $(r, \theta, \phi)$ .

### 2. ORBIT EQUATION FOR GENERAL STATIC SPHERICALLY SYMMETRIC METRICS

The general static spherically-symmetric metric has the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \quad (5)$$

Proceed as in the case of the derivation of the effective potential and the orbit equation for the Schwarzschild metric (set  $\theta = \pi/2$ , denote by  $E = A(r)\dot{t}$  the conserved energy (per unit rest mass) conjugate to the cyclic variable  $t$ , and by  $L = r^2\dot{\phi}$  the angular momentum (per unit rest mass) conjugate to the cyclic variable  $\phi$ ).

(a) Show that the orbit equation for  $r = r(\phi)$  is

$$\tilde{B}(u)(u')^2 + u^2 = \frac{\epsilon}{L^2} + \frac{E^2}{L^2} \frac{1}{\tilde{A}(u)} , \quad (6)$$

where  $u = 1/r$ ,  $u' = du/d\phi$ ,  $\tilde{A}(u) = A(r)$  etc., and where  $\epsilon = -1, +1, 0$  for timelike, spacelike and null geodesics respectively.

(b) Show that, in terms of a turning point  $u_m$  (minimum or maximum of  $u = 1/r$  or  $r$ ) of the trajectory ( $u' = 0$ ), the orbit equation for lightrays can be written as

$$\frac{d\phi}{du} = \pm \tilde{B}(u)^{1/2} \left[ \frac{\tilde{A}(u_m)}{\tilde{A}(u)} u_m^2 - u^2 \right]^{-1/2} . \quad (7)$$

(c) Check that for  $A(r) = B(r) = 1$  this equation describes a straight line.