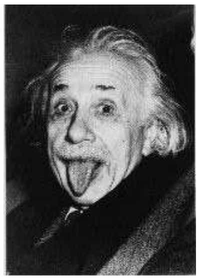
GR Assignments 04



1. Tensor Analysis II: the Covariant Derivative

   The *covariant derivative* ∇_μ is the tensorial generalisation of the partial derivative ∂_μ, i.e. it is such that the covariant derivative of a tensor is again a tensor. More precisely, the covariant derivative of a (p,q)-tensor is then a $(p,q+1)$-tensor because it has one more lower (covariant) index. Since the partial derivative $\partial_\mu f$ of a scalar f (a $(0,0)$-tensor) is a covector (a $(0,1)$-tensor), one sets $\nabla_\mu f = \partial_\mu f$. However, as we have seen, the partial derivative $\partial_\mu V^\nu$ of a vector V^ν (a $(1,0)$-tensor) is *not* a $(1,1)$-tensor. This can be rectified by defining the covariant derivative of a vector to be

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad . \tag{1}$$

   It can be checked that this is indeed a tensor, the non-tensorial nature of the partial derivative cancelling exactly against that of the Christoffel symbols. A smilar story holds for covectors: the partial derivative $\partial_\mu A_\nu$ of a $(0,1)$-tensor (covector) is *not* a tensor. This can be cured in the same way as for vectors, and one can check that

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{2}$$

   is indeed a $(0,2)$-tensor. The action of ∇_μ on vectors and covectors can be extended to arbitrary (p,q)-tensors. For instance, for a $(0,2)$-tensor $B_{\mu\nu}$ one has

$$\nabla_\lambda B_{\mu\nu} = \partial_\lambda B_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} B_{\mu\rho} \quad . \tag{3}$$

   (a) An alternative way to arrive at (2) is to demand the *Leibniz rule* for the covariant derivative of a product of tensors: deduce (2) from (1) using the fact that $A_\nu V^\nu$ is a scalar for any vector V^ν, so that $\nabla_\mu(A_\nu V^\nu) = \partial_\mu(A_\nu V^\nu)$. and using the Leibniz rule for ∂_μ (i.e. $\partial_\mu(A_\nu V^\nu) = (\partial_\mu A_\nu)V^\nu + A_\nu \partial_\mu V^\nu$) and ∇_μ.

   (b) Check that, even though $\partial_\mu A_\nu$ is *not* a tensor, the *curl* (or *rotation*) $\partial_\mu A_\nu - \partial_\nu A_\mu$ *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{4}$$

This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.

(c) Show that (3) implies that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

2. Radial Fall & the repulsive Reissner-Nordstrøm Core

The Reissner-Nordstrøm metric

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})dt^2 + (1 - \frac{2m}{r} + \frac{q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \tag{5}$$

is a solution to the coupled Einstein-Maxwell equations describing the gravitational field of a spherically symmetric electrically charged star ($m \sim$ mass, $q \sim$ charge). We will assume $m^2 > q^2$.

You have already determined the effective potential for this metric in exercise 03.2. Now consider the special case of radialy infalling (angular momentum $L = 0$) massive particles.

(a) Show that for $q \neq 0$ radially infalling (and electrically neutral) massive particles cannot reach $r = 0$ and are reflected by the Reissner-Nordstrøm metric (there is thus a *repulsive* gravitational force (anti-gravity ...) at the core of the Reissner-Nordstrøm solution) and compare and contrast this with the behaviour of radially infalling particles in the Schwarzschild geometry.

(b) Determine the turning point r_{min} of the trajectory of a radially infalling massive particle starting out at rest at infinity (this gives a condition on E or E_{eff}) and show that $r_{min} < r_-$, where r_- is the smaller of the two roots of the equation $f(r) = 1 - 2m/r + q^2/r^2 = 0$.