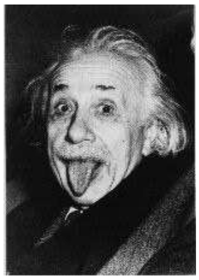

GR Assignments 03

1. Tensor Analysis I: Tensor Algebra

   (a) Show that the partial derivative $\partial_\mu f(x) = \partial f(x)/\partial x^\mu$ of a scalar $f(x)$ transforms like (and hence is) a covector.

   Remark: The partial derivative of a higher-rank tensor is *not* a tensor (you can easily check this yourself), forcing us to introduce a suitable tensorial generalisation of the partial derivative, the *covariant derivative.*

   (b) Let $A_{\mu\nu}$ be a $(0,2)$-tensor and B^μ a $(1,0)$-tensor (a vector). Show that $A_{\mu\nu}B^\nu$ is a co-vector (i.e. transforms like a co-vector) and that $A_{\mu\nu}B^\mu B^\nu$ is a scalar.

   Remark: In particular, given a metric $g_{\mu\nu}$, the *scalar product* $g_{\mu\nu}V^\mu W^\nu$ of two *vectors* V and W is indeed a *scalar* in the tensorial sense.

   (c) Let $V^\mu(x)$ be a vector field and denote by $\partial_\mu = \partial/\partial x^\mu$ the partial derivatives. Show that the first-order linear differential operator

   $$V(x) = V^\mu(x)\ \partial_\mu \tag{1}$$

   is invariant under coordinate transformations. Analagously, let $A_\mu(x)$ be a covector. Show that

   $$A(x) = A_\mu(x) dx^\mu \tag{2}$$

   is invariant under coordinate transformations.

   Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V. V can be expanded in a basis ∂_μ, and its components with respect to this basis are the V^μ. If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.

2. The Effective Geodesic Potential

   Generalising the discussion in section 24.1 of the lecture notes, and following Remark 4 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{3}$$

   (this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).