GR Assignments 09

1. Properties of the Riemann Curvature Tensor

   In the course, we defined the Riemann curvature tensor via the commutator of covariant derivatives,

   $$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu}V^\sigma \quad . \tag{1}$$

   and we defined its contractions, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\ \alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries

   $$\begin{aligned} &(I) \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \qquad (II) \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \\ &(III) \quad R_{\alpha[\beta\gamma\delta]} = 0 \ \Leftrightarrow \ R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \end{aligned} \tag{2}$$

   [see section 7.3 of the lecture notes for proofs and make sure that you understand the details!]

   (a) Show that the above symmetries imply the property

   $$(IV) \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \quad . \tag{3}$$

   (b) Show that symmetry (IV) implies that the Ricci tensor is symmetric.

   (c) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity

   $$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0 \quad . \tag{4}$$

   [If you like, you can check this explicitly by writing out all the commutators.] Show that this, together with the definition (1), implies the *Bianchi identity*

   $$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in } (\alpha, \beta, \gamma) = 0 \tag{5}$$

   (d) By contracting the Bianchi identity over the indices (μ, β) (multiplication by $g^{\mu\beta}$) and (ν, α) (multiplication by $g^{\nu\alpha}$), deduce the identity (*contracted Bianchi identity*)

   $$\nabla_\alpha(2R^\alpha_{\ \gamma} - \delta^\alpha_{\ \gamma}R) = 0 \tag{6}$$

   and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$.

   Remark: In all these equations indices are lowered and raised with the metric and its inverse: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}R^\lambda_{\ \beta\gamma\delta}$, $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$, $R^\alpha_{\ \gamma} = g^{\alpha\beta}R_{\beta\gamma}$ etc.