GR ASSIGNMENTS 08

1. ON THE KLEIN-GORDON FIELD IN A CURVED SPACE-TIME

   The energy-momentum tensor of a real (free, massice) scalar field ϕ is (up to a convention-dependent constant)

   $$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{1}$$

   where

   $$L = -\tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{2}$$

   is the Lagrangian. Show that $T_{\alpha\beta}$ is conserved when ϕ is a solution to the Klein-Gordon equation of motion,

   $$\left(\Box - m^2\right)\phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ , \tag{3}$$

   where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$.

2. ON THE MAXWELL EQUATIONS IN CURVED SPACE-TIME
   (SEE SECTION 5.3 OF THE LECTURE NOTES FOR DETAILS AND NOTATION)

   (a) Derive the vacuum Maxwell equations

   $$\nabla_\mu F^{\mu\nu} = 0 \tag{4}$$

   by variation of the action

   $$S = \tfrac{1}{4}\int \sqrt{g}d^4x\ g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} \ . \tag{5}$$

   with respect to the gauge field A_μ.

   Remark: You can write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ either in terms of partial or in terms of covariant derivatives. In the former case, you will need to use the equation $\nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu})$ for the covariant divergence in terms of $\sqrt{g}$ etc. which you derived before. In the latter case, you can use the fact that $\sqrt{g}$ is covariantly constant ($\nabla_\mu\sqrt{g} = 0$).

(b) Use the Maxwell equations (with a source),

$$\nabla_\mu F^{\mu\nu} = -J^\nu \quad , \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0 \;\Leftrightarrow\; \nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0 \tag{6}$$

and the formula

$$T^{\mu\nu} = F^\mu_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho}F^{\lambda\rho} \tag{7}$$

for the electromagnetic energy-momentum tensor, to deduce the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \quad . \tag{8}$$

Remark: This is a tensorial equation. For such calculations you should just use the properties of the covariant derivative and **not** write out the covariant derivative in terms of the non-tensorial Christoffel symbols and partial derivatives.

Hint: Instead of embarking blindly on this calculation, remind yourself first how to do the calculation in Minkowski space. Exactly the same procedure should then work in general. If done correctly, this should be a four-line calculation.

3. TENOSR ANALYSIS V: PARALLEL TRANSPORT

In the course, we have discussed the notion of *covariant derivative along a curve*, denoted by $\nabla_\tau = \dot{x}^\mu \nabla_\mu$ (in the course) or $D/D\tau$ (in the lecture notes). A closely related concept is that of *Parallel Transport* (see section 4.8 of the notes). Note that, in a general (curved) metric space time, it does not make sense to ask if two vectors defined at points x and y are parallel to each other or not. However, given a metric and a curve connecting these two points, one can compare the two by dragging one along the curve to the other using the covariant derivative. We say that a tensor $T^{\cdots}_{\cdots}$ is *parallel transported along the curve* $x^\mu(\tau)$ if

$$\frac{DT^{\cdots}_{\cdots}}{D\tau} = 0 \quad . \tag{9}$$

In a locally inertial coordinate system along the curve, this condition reduces to $dT/d\tau = 0$, i.e. to the statement that the tensor does not change along the curve. Thus the above is indeed an appropriate tensorial generalisation of the intuitive notion of parallel transport to a general space-time (or metric space). The following are all 1-line calculations:

(a) Show that the tangent vector $\dot{x}^\mu$ is parallel transported along the curve $x^\mu(\tau)$ precisely when the curve is a geodesic (for this reason, geodesics are occasionally also called *auto-parallels*).

(b) Show that the length^2 $g_{\mu\nu}V^\mu V^\nu$ of a parallel transported vector V^μ is constant along the curve.

(c) Let $x^\mu(\tau)$ be a geodesic, and V^μ a parallel transported vector. Show that the scalar product $g_{\mu\nu}\dot{x}^\mu V^\nu$ is constant along the curve.