# GR Assignments 06

1. Rindler Coordinates and the Schwarzschild Geometry near $r = r_S$

   Recall that in Rindler coordinates $(\rho, \eta)$,

   $$x^0 = \rho \sinh \eta \qquad x^1 = \rho \cosh \eta \ . \tag{1}$$

   the 2-dimensional Minkowski metric $ds^2 = -(dx^0)^2 + (dx^1)^2$ takes the form

   $$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \ . \tag{2}$$

   The purpose of this exercise is to show that the $(t, r)$-part of the geometry of the Schwarzschild metric near the Schwarzschild radius $r = r_S \equiv 2m$ has exactly the above form (with $r \to r_S \Leftrightarrow \rho \to 0$) establishing that the geometry is non-singular at $r = r_S$ and providing insight into the physics of the Schwarzschild metric and coordinates.

   (a) Show first that the curves $\rho = \rho_0 =$ const. are the worldlines of Minkowski observers with constant acceleration $\mathsf{a} = 1/\rho_0$. [Recall that for a proper-time normalized 4-velocity $u^\alpha = \dot{x}^\alpha$ with acceleration $a^\alpha = \dot{u}^\alpha$ one has $u^\alpha u_\alpha = -1, u^\alpha a_\alpha = 0, a^\alpha a_\alpha \equiv \mathsf{a}^2 > 0$.]

   (b) Now consider the near-$r_S$ geometry of the Schwarzschild metric, defined by approximating $(1 - 2m/r)$ and its inverse by

   $$1 - \frac{2m}{r} = \frac{r - 2m}{r} \approx \frac{r-2m}{2m} \quad , \quad \left(1 - \frac{2m}{r}\right)^{-1} \approx \frac{2m}{r - 2m} \ . \tag{3}$$

   Introduce a new radial coordinate $\rho$ as the proper radial distance ($d\rho = \sqrt{2m/(r-2m)}dr$) from $r = r_S$, and the rescaled time-coordinate $\eta = t/4m$. Show that $\rho^2 = 8m(r - 2m)$ and that the $(t, r)$-part of the near-$r_S$ Schwarzschild metric takes precisely the above Rindler form (2).

2. Kruskal Coordinates for the Schwarzschild Space-Time

   To get to a completely non-singular form of the Schwarzschild metric (for any $r \neq 0$, not just near $r = r_S$), introduce the so-called *Kruskal coordinates*

   $$\begin{aligned} X &= \tfrac{1}{2}(\mathrm{e}^{(t + r^*)/4m} + \mathrm{e}^{-(t - r^*)/4m}) \\ T &= \tfrac{1}{2}(\mathrm{e}^{(t + r^*)/4m} - \mathrm{e}^{-(t - r^*)/4m}) \ . \end{aligned} \tag{4}$$

where the *tortoise coordinate* $r^*$ is defined by $r^* = r + 2m\log(r/2m - 1)$. [I will explain the motivation for this choice of coordinates in detail next week. For now just recall that $r^*$ appeared in the solution for radial lightrays in Assignments 04.]

(a) Show that in terms of these coordinates the metric is

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}(-dT^2 + dX^2) + r^2 d\Omega^2 \quad , \tag{5}$$

where $r = r(T, X)$ is implicitly given by

$$X^2 - T^2 = (r/2m - 1)\mathrm{e}^{r/2m} \quad . \tag{6}$$

(b) Express the locations of the Schwarzschild radius ($r = 2m$) and the central singularity ($r = 0$) in terms of $X$ and $T$.

3. Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

In the Schwarzschild coordinates $(t, r)$, the Schwarzschild metric has the standard form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \qquad\qquad f(r) = 1 - \frac{2m}{r} \quad . \tag{7}$$

(a) Show that the metric

$$ds^2 = -f(r)dT^2 + 2C(r)dTdr + f(r)^{-1}(1 - C(r)^2)dr^2 + r^2 d\Omega^2 \tag{8}$$

is equivalent to the Schwarzschild metric for *any* function $C(r)$. [**Hint:** Begin with (7) and consider the coordinate transformation $T(t, r) = t + \psi(r)$.]

(b) Now choose $C(r)$ such that $g_{rr} = 1$ (Painlevé-Gullstrand (PG) coordinates). Write down the resulting metric and show that it is completely non-singular for all $r > 0$ (in particular for $r \to 2m$), i.e. show that the metric coefficients are bounded and the determinant is non-zero.

(c) Show that the choice $C(r) = 1$ gives rise to the metric in Eddington-Finkelstein coordinates (with $T \equiv u = t + r^*$).

**Optional Further Exercises:**

Test your understanding/knowledge of GR (solutions will *not* be provided).

The metric in PG coordinates is related to timelike geodesics in the same way as the metric in Eddington-Finkelstein coordinates is related to null geodesics. To see this, consider the field of normal vectors $u_\alpha = -\partial_\alpha T$ orthogonal to the surfaces of constant $T$ (in Scharzschild coordinates $x^\alpha = (t, r, \ldots)$).

(d) Show that $u^\alpha u_\alpha = -1$. Then show that in general the two properties $u^\alpha u_\alpha =$ const and $u_\alpha = -\partial_\alpha T$ imply that $u^\alpha$ is geodesic, i.e. $u^\beta \nabla_\beta u^\alpha = 0$.

(e) Show that the geodesics $x^\alpha(\tau)$ to which the $u^\alpha$ are tangent ($u^\alpha = \dot{x}^\alpha$) are radial geodesics ($L = 0$) with proper time $\tau = T$ and energy $E = 1$ (corresponding to observers that would have started off at rest at $r = \infty$).