GR Assignments 05

1. Orbit Equation for general static spherically symmetric Metrics

   The general static spherically-symmetric metric has the form

   $$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{1}$$

   Proceeding as in the case of the derivation of the effective potential and the orbit equation for the Schwarzschild metric (set $\theta = \pi/2$, denote by $E = A(r)\dot{t}$ the conserved energy (per unit rest mass) conjugate to the cyclic variable t, and by $L = r^2\dot{\phi}$ the angular momentum (per unit rest mass) conjugate to the cyclic variable ϕ), show that the orbit equation for $r = r(\phi)$ is

   $$\tilde{B}(u)(u')^2 + u^2 = \frac{\epsilon}{L^2} + \frac{E^2}{L^2}\frac{1}{\tilde{A}(u)} \quad , \tag{2}$$

   where $u = 1/r$, $u' = du/d\phi$, $\tilde{A}(u) = A(r)$ etc., and where $\epsilon = -1, +1, 0$ for timelike, spacelike and null geodesics respectively.

2. Radial Fall & the repulsive Reissner-Nordstrøm Core

   The Reissner-Nordstrøm metric

   $$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})dt^2 + (1 - \frac{2m}{r} + \frac{q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \quad . \tag{3}$$

   is a solution to the coupled Einstein-Maxwell equations describing the gravitational field of a spherically symmetric electrically charged star ($m \sim$ mass, $q \sim$ charge). Show that the effective geodesic potential equation for radial (angular momentum $L = 0$) timelike geodesics is simply $\dot{r}^2/2 + V_{\mathit{eff}}(r) = \mathcal{E}$ where

   $$V_{\mathit{eff}}(r) = -\frac{m}{r} + \frac{q^2}{2r^2} \quad , \quad \mathcal{E} = \tfrac{1}{2}(E^2 - 1) \quad . \tag{4}$$

   Conclude that for $q \neq 0$ even radially infalling (and electrically neutral) particles cannot reach $r = 0$ and are reflected by the Reissner-Nordstrøm metric (there is thus a *repulsive* gravitational force (anti-gravity ...) at the core of the Reissner-Nordstrøm solution).

   Hint: Note that this effective potential is *identical* to the effective potential of a Newtonian particle in the potential $-m/r$ with angular momentum $L = q$.

3. Tensor Analysis IV: the Riemann Curvature Tensor

(a) One of the characteristic properties of the covariant derivative ∇_μ is that *second covariant derivatives acting on scalars commute*,

$$[\nabla_\mu, \nabla_\nu]\phi \equiv (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\phi = 0 \ . \tag{5}$$

Show that this follows from the symmetry of the Christoffel symbols.

(b) This, however, is no longer true when covariant derivatives act on tensors of higher rank. In general, *second covariant derivatives acting on vectors do not commute*. However, the commutator $[\nabla_\mu, \nabla_\nu]V^\lambda$ turns out to depend only linearly on V and not on its derivatives, i.e. one has

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu} V^\sigma \tag{6}$$

for some collection of objects $R^\lambda_{\ \sigma\mu\nu}$. Since everything on the left and V on the right hand side of this equation are tensors, also the $R^\lambda_{\ \sigma\mu\nu}$ are the components of a tensor, namely the famous *Riemann(-Christoffel) Curvature Tensor*. By explicit calculation show that the Riemann tensor is given by

$$R^\lambda_{\ \sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\ \sigma\nu} - \partial_\nu \Gamma^\lambda_{\ \sigma\mu} + \Gamma^\lambda_{\ \mu\rho}\Gamma^\rho_{\ \nu\sigma} - \Gamma^\lambda_{\ \nu\rho}\Gamma^\rho_{\ \mu\sigma} \ . \tag{7}$$

Remark: Note that this tensor is clearly zero for the Minkowski metric written in Cartesian coordinates. (Why?) Hence it is also zero for the Minkowski metric written in any other coordinate system. (Why?) Conversely, therefore, if one has a metric for which at least one component of the Riemann tensor is non-zero, this metric cannot be equivalent to the Minkowski metric via a coordinate transformation.