

## GR ASSIGNMENTS 04

### 1. STATIONARY OBSERVERS AND LIGHTRAYS IN THE SCHWARZSCHILD GEOMETRY

- (a) Consider a stationary observer (sitting at fixed values of  $(r, \theta, \phi)$ ) in the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) . \quad (1)$$

Determine his worldline 4-velocity  $u^\alpha = dx^\alpha/d\tau$  and the acceleration  $a^\alpha = \nabla_\tau u^\alpha \equiv u^\beta \nabla_\beta u^\alpha$  and calculate  $g_{\alpha\beta} a^\alpha a^\beta$ .

- (b) Show that radial lightrays ( $(\theta, \phi)$  constant) satisfy  $dt/dr = \pm(1 - 2m/r)^{-1}$  and solve this equation for  $t = t(r)$ .

### 2. TENSOR ANALYSIS III: THE COVARIANT DIVERGENCE

An extremely useful identity for the variation (in particular, the derivative) of the determinant  $g := |\det g_{\mu\nu}|$  of the metric is

$$g^{-1} \delta g = g^{\mu\nu} \delta g_{\mu\nu} \quad g^{-1} \partial_\lambda g = g^{\mu\nu} \partial_\lambda g_{\mu\nu} . \quad (2)$$

One proof of this result, based on the explicit expansion of a determinant in terms of minors,  $g = \sum_\nu (-1)^{\mu+\nu} g_{\mu\nu} |m_{\mu\nu}|$  is described in the lecture notes (section 4.6). Another easy proof can be deduced from the fundamental matrix identity  $\det M = \exp \operatorname{tr} \log M$  (see the supplement to these exercises).

- (a) Use (2) to show that the contracted Christoffel symbol  $\Gamma^\mu_{\mu\lambda}$  (summation over the index  $\mu$ ) can be calculated by the simple formula

$$\Gamma^\mu_{\mu\lambda} = g^{-1/2} \partial_\lambda g^{1/2} . \quad (3)$$

- (b) Show that this implies that the covariant divergence of a vector (current)  $J^\mu$  and an anti-symmetric tensor  $F^{\mu\nu} = -F^{\nu\mu}$  can be written as

$$\nabla_\mu J^\mu = g^{-1/2} \partial_\mu (g^{1/2} J^\mu) \quad \nabla_\mu F^{\mu\nu} = g^{-1/2} \partial_\mu (g^{1/2} F^{\mu\nu}) . \quad (4)$$