

## GR ASSIGNMENTS 03

### Tensor Analysis II: The Covariant Derivative

The *covariant derivative*  $\nabla_\mu$  is the tensorial generalisation of the partial derivative  $\partial_\mu$ , i.e. it is such that the covariant derivative of a tensor is again a tensor. More precisely, the covariant derivative of a  $(p, q)$ -tensor is then a  $(p, q + 1)$ -tensor because it has one more lower (covariant) index. Since the partial derivative  $\partial_\mu f$  of a scalar  $f$  (a  $(0, 0)$ -tensor) is a covector (a  $(0, 1)$ -tensor), one sets  $\nabla_\mu f = \partial_\mu f$ . However, as we have seen, the partial derivative  $\partial_\mu V^\nu$  of a vector  $V^\nu$  (a  $(1, 0)$ -tensor) is *not* a  $(1, 1)$ -tensor. This can be rectified by defining the covariant derivative of a vector to be

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda . \quad (1)$$

It can be checked that this is indeed a tensor, the non-tensorial nature of the partial derivative cancelling exactly against that of the Christoffel symbols. A similar story holds for covectors: the partial derivative  $\partial_\mu A_\nu$  of a  $(0, 1)$ -tensor (covector) is *not* a tensor. This can be cured in the same way as for vectors, and one can check that

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \quad (2)$$

is indeed a  $(0, 2)$ -tensor. The action of  $\nabla_\mu$  on vectors and covectors can be extended to arbitrary  $(p, q)$ -tensors. For instance, for a  $(0, 2)$ -tensor  $B_{\mu\nu}$  one has

$$\nabla_\lambda B_{\mu\nu} = \partial_\lambda B_{\mu\nu} - \Gamma^\rho_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho_{\lambda\nu} B_{\mu\rho} . \quad (3)$$

1. An alternative way to arrive at (2) is to demand the *Leibniz rule* for the covariant derivative of a product of tensors: deduce (2) from (1) using the fact that  $A_\nu V^\nu$  is a scalar for any vector  $V^\nu$ , so that  $\nabla_\mu(A_\nu V^\nu) = \partial_\mu(A_\nu V^\nu)$ . and using the Leibniz rule for  $\partial_\mu$  (i.e.  $\partial_\mu(A_\nu V^\nu) = (\partial_\mu A_\nu)V^\nu + A_\nu \partial_\mu V^\nu$ ) and  $\nabla_\mu$ .
2. Check that, even though  $\partial_\mu A_\nu$  is *not* a tensor, the *curl* (or *rotation*)  $\partial_\mu A_\nu - \partial_\nu A_\mu$  *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu . \quad (4)$$

This provides an alternative argument for the fact that  $\partial_\mu A_\nu - \partial_\nu A_\mu$  is a tensor.

3. Show that (3) implies that the covariant derivative of the metric is zero,  $\nabla_\lambda g_{\mu\nu} = 0$ .