GR ASSIGNMENTS 02

1. GEODESICS

   (a) GEODESICS AND EULER-LAGRANGE EQUATIONS: Show that the Euler-Lagrange equations

   $$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \ , \tag{1}$$

   for the Lagrangian

   $$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2}$$

   are the geodesic equations

   $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ , \tag{3}$$

   where, as usual,

   $$\begin{aligned} \Gamma^\mu_{\ \nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} \\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_\lambda + g_{\mu\lambda},_\nu - g_{\nu\lambda},_\mu) \ . \end{aligned} \tag{4}$$

   (b) $\mathcal{L}$ IS A CONSTANT OF MOTION: Show that $\mathcal{L}$ is constant along any geodesic, i.e. that

   $$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0 \tag{5}$$

   for $x^\mu(\tau)$ a solution of the geodesic equation.

   (c) GEODESICS ON THE TWO-SPHERE S^2: The metric of a 2-sphere with radius R is

   $$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{6}$$

   Use (4) to calculate all its Christoffel symbols, show that the geodesic equations agree with the Euler-Lagrange equations of the Lagrangian

   $$\mathcal{L} = \tfrac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ , \tag{7}$$

   and show that the great circles (longitudes) $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ satisfy the geodesic equation.

2. TENSOR ANALYSIS I: TENSOR ALGEBRA

This will be the first in an important series of exercises related to tensor analysis. Please consult the lecture notes for more detailed information!

Tensors are objects that transform in a particularly simple (multi-linear) way under coordinate transformations $x^\mu \to y^{\alpha'}$. The simplest examples are *scalars* (functions) $f(x)$ that don't transform at all, $f'(y) = f(x)$, *vectors* $V^\mu(x)$ that transform as

$$V'^{\alpha'}(y(x)) = \frac{\partial y^{\alpha'}}{\partial x^\mu} V^\mu(x) \quad , \tag{8}$$

and *covectors* $A_\mu(x)$ that transform inversely to a vector, i.e. as

$$A'_{\alpha'}(y(x)) = \frac{\partial x^\mu}{\partial y^{\alpha'}} U_\mu(x) \quad . \tag{9}$$

A (p,q)-tensor is an object $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}$ with p upper (contravariant) indices and q lower (covariant) indices that transforms like a product of p vectors and q covectors. Thus, in particular, the metric $g_{\mu\nu}$ is a $(0,2)$-tensor (or covariant 2-tensor). Tensor algebra works just as for Lorentz-tensors in special relativity, i.e. one can add and multiply tensors in the usual way.

(a) Show that the partial derivative $\partial_\mu f(x) = \partial f(x)/\partial x^\mu$ of a scalar $f(x)$ transforms like (and hence is) a covector.

(b) Let $A_{\mu\nu}$ be a $(0,2)$-tensor and B^μ a $(1,0)$-tensor (a vector). Show that $A_{\mu\nu}B^\nu$ is a co-vector (i.e. transforms like a co-vector) and that $A_{\mu\nu}B^\mu B^\nu$ is a scalar.

Remark: In particular, given a metric $g_{\mu\nu}$, the *scalar product* $g_{\mu\nu}V^\mu W^\nu$ of two *vectors* V and W is indeed a *scalar* in the tensorial sense.

(c) Let $V^\mu(x)$ be a vector field and denote by $\partial_\mu = \partial/\partial x^\mu$ the partial derivatives. Show that the first-order linear differential operator

$$V(x) = V^\mu(x)\, \partial_\mu \tag{10}$$

is invariant under coordinate transformations. Analagously, let $A_\mu(x)$ be a covector. Show that

$$A(x) = A_\mu(x) dx^\mu \tag{11}$$

is invariant under coordinate transformations.

Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V. V can be expanded in a basis ∂_μ, and its components with respect to this basis are the V^μ. If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.