Field Theory on Schrödinger Space-Time

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The Schrödinger space-time plays an important role in the context of non-relativistic holography. We discuss causal structure properties of the Schrödinger space-time by probing it with point particles as well as with scalar fields. We show that even though the causal structure seen by point particles is almost pathological (absence of a time function) this is not so for the scalar fields. We highlight Galilean-like causal structure properties.

1 Introduction

Starting with [1–3] one has tried to construct holographic techniques to study systems with non-relativistic symmetries. Such systems typically occur in the area of condensed matter physics, e.g. in the context of quantum critical points [4, 5]. Typically one is interested in some strongly coupled scale invariant field theory that describes some critical point. These theories arise as effective infrared descriptions of some physical system. Typically these scale invariant theories are invariant under Lifshitz, Schrödinger or the full relativistic conformal symmetry group. Often these strongly coupled scale invariant field theories describing some critical point belong to a universality class. One assumes that the universality class also contains a gravitational theory that admits a holographic dual. The goal is then to construct holographic techniques for space-times with non-relativistic isometry groups such as Lifshitz and Schrödinger groups.

As a first step in realising these goals we study quantum field theory on a fixed Schrödinger background and focus on its causal structure properties as these will turn out to be somewhat unusual. Nonetheless we will argue that it is possible to have a well-defined quantum field theory on a fixed Schrödinger background.

2 Causal properties of Schrödinger space-times

In this section we will discuss the causal structure associated with point particles moving along future directed causal curves in the Schrödinger space-time. In global coordinates [6] the Schrödinger metric reads

$$ds^2 = -\left(\frac{\beta^2}{R^4} + \frac{\omega^2}{R^2}(\vec{X}^2 + R^2)\right) dT^2 + \frac{1}{R^2}\left(-2dTdV + d\vec{X}^2 + dR^2\right) . \quad (1)$$

When the parameter $\beta = 0$ this becomes the metric of AdS space-time in plane wave coordinates. When $\omega = 0$ we obtain the metric in Poincaré coordinates. We will collect here some basic causal structure properties of this space-time indicating which properties hold only for $\beta \neq 0$ (Schrödinger) and which also hold for $\beta = 0$ (AdS). The definitions used below follow [7, 8]. Global coordinate time T is a globally defined smooth function. The vector field ∂_T is an everywhere timelike Killing vector, which provides a

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time orientation. The function T is strictly increasing along any future-directed timelike curve and non-decreasing along any future-directed null curve. The last fact implies that the space-time is chronological.

A time function is a globally defined continuous function that is strictly increasing along all future-directed causal curves. It therefore provides an ordering, as all causally related events can then be labeled by different values of T. The existence of a time-function is equivalent to the space-time being stably causal, and this in turn is equivalent to the existence of a (not necessarily the same) globally defined function whose gradient is everywhere timelike [7, 8]. Hence T is not a time function, neither for Schrödinger nor for AdS. So what about stable causality of these space-times?

An important difference between AdS and Schrödinger space-times is that for AdS we can construct time functions, e.g. the time coordinate of the usual global AdS coordinate system, whereas for Schrödinger we cannot. Therefore Schrödinger space-times are not stably causal.

The manner in which T fails to be a time function is very precise. The only events that are not distinctly labeled by T lie on so-called lightlike lines.[1] Lightlike lines are always null geodesics but the converse is generally not true. In space-times such as Minkowski and AdS all null geodesics are lightlike lines. In our context the lightlike lines are given by the following future-directed null geodesics along which T remains constant

$$\gamma(\lambda) = (T_0, V(\lambda), R_0, \vec{X}_0)\,, \tag{2}$$

where $V(\lambda)$ is a monotonically increasing function of λ.

This being said the truly dramatic effect of having a non-zero β is that it makes the space-time non-distinguishing.[2] This has already been proven in [10] for the $z = 3$ Schrödinger space-time and the possible connection of this property with a Galilean-like causal structure was noted in [11]. The proof of [10] is based on the existence of a causal curve that connects any two points whose time interval is infinitesimally small.

A curve similar to the one used in [10] to prove the non-distinguishing character of the Schrödinger space-time can be used to explicitly find the chronological future (past), $I^{\pm}(p_0)$, of any point $p_0 = (T_0, V_0, R_0, \vec{X}_0)$. It turns out to be the set of all points with $T > T_0$ ($T < T_0$). Therefore, for any point p_0 one has the decomposition

$$\mathsf{Sch} = I^{-}(p_0) \cup \Sigma_{T_0} \cup I^{+}(p_0) \tag{3}$$

of the Schrödinger space-time. Since all points on a constant time slice share the same future and past, the space-time is in a sense "maximally non-distinguishing".

This is strongly reminiscent of a Galilean causal structure and Galilean relativity. In order to sharpen this analogy, we need an appropriate notion of spacelike separation. We will call two points x and x' *spacelike separated* if there is no causal curve connecting them. It is perhaps worth pointing out that this notion of spacelike separation does not imply that two points are spacelike separated when they can be connected by a spacelike geodesic. According to this definition spacelike separated points necessarily lie on an equal-time T slice Σ_T.

This appears to be completely Galilean, since in Galilean relativity any two non-simultaneous events can be connected by the worldline of a (sufficiently fast moving) particle, and the only events for which no such curve exists are those that are simultaneous. However, the novel and non-Galilean feature of the causal structure of Schrödinger space-times is the presence of lightlike lines. Indeed, on a Schrödinger space-time all points with the same value of T are either spacelike separated or separated by a lightlike line and conversely all points that are either spacelike separated or separated by a lightlike line lie on an equal time T surface.

This Galilean-like structure is preserved by the subgroup

$$(T', V', R', \vec{X}') = (T'(T), V'(T, V, R, \vec{X}), R'(T, R, \vec{X}), \vec{X}'(T, R, \vec{X})) \tag{4}$$

[1] A lightlike line is an achronal inextendible causal curve [9]. A set S is called achronal resp. acausal if no two distinct points of S can be connected by a timelike resp. causal curve.

[2] A space-time is called non-distinguishing if there exist two distinct points that have identical past and future.

of the full group of space-time diffeomorphisms. Indeed, any set of coordinates $(T', V', R', \vec{X}')$ obtained by acting on the global coordinates $(T, V, R, \vec{X})$ with such a diffeomorphism is such that T', the new time coordinate, labels surfaces of spacelike and lightlike line separated events while any new V' coordinate parametrises the lightlike lines. The normal to a constant T' slice $\Sigma_{T'}$ is proportional to the null Killing vector N, and the (degenerate) induced metric on $\Sigma_{T'}$ agrees with the Galilean metric measuring the distance between simultaneous (spacelike) separated events. This special class of diffeomorphisms consists precisely of the double foliation preserving diffeomorphisms discussed in a related context in [12]. Here the double foliation refers to the foliations associated with the equal time surfaces and the lightlike lines.

3 Scalar field theory

The causal structure properties of the Schrödinger space-time may cast doubt on there being a well-defined quantum field theory. Potential problems deriving from the absence of a time function and the non-distingushing nature of the space-time could arise with time ordering, time evolution or predictability. In this section we will study the causal structure of Schrödinger space-times as seen by scalar field probes and show that, even though the causal structure seen by point particles is close to pathological, this is not so from the point of view of the scalars.

The action for a massive complex scalar field ϕ is

$$S = -\int d^{d+3}x\sqrt{-g}\left(\partial_\mu\phi^*\partial^\mu\phi + m_0^2\phi^*\phi\right) + \ldots\,, \tag{5}$$

where m_0 is a mass parameter and the dots refer to boundary terms. We will consider scalar fields ϕ that are eigenstates of the central element ∂_V of the Schrödinger algebra, i.e.

$$\phi(T, V, R, \vec{X}) = e^{-imV}\psi(T, R, \vec{X})\,, \tag{6}$$

in which $m \neq 0$, and we will decompose solutions to the scalar field equation formally as

$$\phi = \sum_M a_M u_M\,, \tag{7}$$

where the $u_M(T, V, R, \vec{X})$ form a complete set of modes with a fixed momentum m in the V direction, $u_M(T, V, R, \vec{X}) = e^{-imV}v_M(T, R, \vec{X})$. These states furnish a unitary irreducible representation of the Schrödinger group with respect to the inner product

$$\langle u_M|u_{M'}\rangle = \frac{i}{2}\int_{\Sigma_T} d\Sigma^\mu u_M^* \overleftrightarrow{\partial_\mu} u_{M'}\,. \tag{8}$$

The $T =$ cst slice Σ_T is a lightlike surface whose normal is $\left(\frac{\partial}{\partial V}\right)^\mu = \delta^\mu_V$. The integration measure is $d\Sigma^\mu = \delta^\mu_V R^{-(d+1)}dRd^d\vec{X}dV$.

In (6) we assume that $m \neq 0$. For modes with $m = 0$ the time-dependence is not fixed by the Klein–Gordon equation. Since these are the modes with zero lightcone momentum, $P_-\phi = 0$, they can be thought of as the precise scalar field counterparts of the lightlike lines discussed in section 3. It turns out that for a free non-interacting theory these modes do not appear in the phase space of the theory. They are zero as a consequence of Hamilton's equations (see [13] for an explanation of this fact in Minkowski space-time with a compact null circle). The problems encountered with the $m = 0$ modes in [14] appear only when one studies loop corrections in an interacting theory. It lies beyond the scope of our work to see if similar problems appear on a Schrödinger space-time.

We next construct the possible Hilbert spaces. To this end we need to obtain the normalisable modes. Normalizable modes are solutions that are regular everywhere in the bulk and that furthermore satisfy the boundary condition that the inner product (8) is time independent. This will be the case provided we have

$$\lim_{\varepsilon\to 0}\int_{R=\varepsilon} R^{-(d+1)}u_M^*\overleftrightarrow{\partial_R}u_{M'}dV d^d\vec{X} = 0\,. \tag{9}$$

This is the condition that the flux of the current $u^*_M \overleftrightarrow{\partial_\mu} u_{M'}$ through the boundary at $R = 0$ vanishes. Imposing this boundary condition requires that ν defined by

$$\nu = \sqrt{\tfrac{(d+2)^2}{4} + m_0^2 + \beta^2\, m^2} \tag{10}$$

is real so that all normalisable modes respect the Breitenlohner–Freedman bound [15].

In global coordinates the normalizable modes[3] are given by

$$\begin{aligned}\phi_\pm &= e^{-imV} \sum_{L,n,k} a^\pm_{L,n,k} v^\pm_{L,n,k} \\ &= e^{-imV} \sum_{L,n,k} C^\pm_{L,n,k} a^\pm_{L,n,k} e^{-iE^\pm_{L,n,k}T} Y_L e^{-\frac{1}{2}\omega|m|(\rho^2+R^2)} \rho^L R^{\Delta_\pm} \times \\ &\quad \times L_n^{L-1+d/2}(\omega|m|\rho^2) L_k^{\pm\nu}(\omega|m|R^2)\,, \end{aligned} \tag{11}$$

where the Y_L with $L = 0, 1, 2, \ldots$ are spherical harmonics on S^{d-1} and the $L_n^{L-1+d/2}$, $L_k^{\pm\nu}$ with $n, k = 0, 1, 2, \ldots$ are generalized Laguerre polynomials. Further we have

$$\Delta_\pm = \frac{d+2}{2} \pm \nu\,, \tag{12}$$

$$E^\pm_{L,n,k} = \text{sign}(m) 2\omega \left(n + k + \tfrac{L}{2} + \tfrac{\Delta_\pm}{2}\right)\,, \tag{13}$$

$$(C^\pm_{L,n,k})^2 = \frac{2(\omega|m|)^{L+\Delta_\pm}}{|m|\pi} \frac{n!k!}{\Gamma(n+L+\frac{d}{2})\Gamma(1+k\pm\nu)}\,. \tag{14}$$

For the minus modes we must assume $0 < \nu < 1$ while for the plus modes we must assume that $\nu > 0$. The cases $\nu = 0, 1, 2, \ldots$ have to be dealt with separately because they involve logarithmic solutions. Here we will always assume that $\nu \neq 0, 1, 2, \ldots$. Upon quantisation the creation and annihilation operators $a^\pm_{L,n,k}$ and $a^{\pm\dagger}_{L,n,k}$ satisfy the commutation relation

$$[a^\pm_{L,n,k}, a^{\pm\dagger}_{L',n',k'}] = \tfrac{1}{2}\text{sign}(m)\delta_{LL'}\delta_{nn'}\delta_{kk'}\,. \tag{15}$$

The sign function on the right hand side of (15) can be understood as follows. The Fock space vacuum $|0\rangle$ is defined by $a^\pm_{L,n,k}|0\rangle = 0$ for $m > 0$ and $a^{\pm\dagger}_{L,n,k}|0\rangle = 0$ for $m < 0$. The interpretation of the latter statement is that $a^{\pm\dagger}_{L,n,k}$ for $m < 0$ is the annihilation operator for the antiparticle making $a^\pm_{L,n,k}$ for $m < 0$ the creation operator for the antiparticle.

For all Hilbert spaces associated with the $\pm$ modes and with $m \neq 0$, denoted by $\mathcal{H}^\pm_m$, there exists a well-posed initial value problem in the sense that given initial data for a scalar field in $\mathcal{H}^\pm_m$ at some time $T = T_0$ it is possible to uniquely predict the future dependence. To see this one just has to note that from $\phi(T = T_0, V, R, \vec{X})$ and the mode decomposition (11) it is possible to read off the coefficients $a_{L,n,k}$ via

$$\langle e^{-imV} v^\pm_{L,n,k} | \phi(T = T_0)\rangle = \text{sign}(m) a^\pm_{L,n,k}\,. \tag{16}$$

Knowing all the $a^\pm_{L,n,k}$ determines the full future dependence of the function ϕ (from (11)). Note that in order to have a well-defined time evolution we only need to specify the values of the field ϕ at time $T = T_0$ and not its first T-derivative.

This structure and property of the initial value problem and time-evolution of scalar fields on Schrödinger space-times is preserved by the foliation-preserving diffeomorphisms (4). In any coordinate system obtained in this way, the Klein-Gordon equation is a first order differential equation in the new time coordinate T', and the evolution of the Klein-Gordon field ϕ is determined by the value of the field on the null surface $\Sigma_{T'}$ (and the momentum in the V'-direction, the mass).

[3] These $\pm$ normalisable modes have also been discussed in global coordinates in [16] and in Poincaré coordinates in [1].

In order to further study the causal structure seen by the scalar fields we look at Green's functions. For this purpose we will first construct the positive and negative frequency Wightman functions, $G^{\pm}(x,x')$. These can be defined for both Hilbert spaces $\mathcal{H}_m^{\pm}$ where the $\pm$ refer to the two different sets of normalisable modes in (11). We will write the expressions for G^+ and G^- on $\mathcal{H}_m^{\pm}$ simultaneously, hoping that this does not cause any confusion. Using the mode decompositions (11) together with an $i\epsilon$ prescription, that guarantees convergence of the sum over the normalizable modes, we obtain

$$G^+(x,x') = \langle 0|\phi(x)\phi^\dagger(x')|0\rangle = \theta(m)\frac{i^{-\Delta_\pm}}{(2\pi)^{\frac{d}{2}}4\pi m}(m\zeta_{-\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(m\zeta_{-\epsilon})e^{im\eta_{-\epsilon}}\,, \tag{17}$$

$$G^-(x,x') = \langle 0|\phi^\dagger(x')\phi(x)|0\rangle = -\theta(-m)\frac{i^{\Delta_\pm}}{(2\pi)^{\frac{d}{2}}4\pi m}(-m\zeta_{+\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(-m\zeta_{+\epsilon})e^{im\eta_{+\epsilon}}\,, \tag{18}$$

where $J_{\pm\nu}$ are Bessel functions and where $\zeta_{\pm\epsilon}$ and $\eta_{\pm\epsilon}$ are

$$\zeta_{\pm\epsilon} = \frac{\omega R R'}{\sin\omega(T-T'\pm i\epsilon)}\,, \tag{19}$$

$$\eta_{\pm\epsilon} = -(V-V') + \frac{\omega(\vec{X}^2+\vec{X}'^2+R^2+R'^2)}{2\tan\omega(T-T'\pm i\epsilon)} - \frac{\omega\vec{X}\cdot\vec{X}'}{\sin\omega(T-T'\pm i\epsilon)}\,. \tag{20}$$

The functions $\zeta_{\pm\epsilon}(x,x')$ and $\eta_{\pm\epsilon}(x,x')$ are for $\epsilon = 0$ invariant under the Schrödinger group when we act on x and x' simultaneously with the same transformation.

Now that we have the two Wightman functions at our disposal we are in a position to compute any Green's function that we are interested in. For example the Feynman propagator is given by (see also [17])

$$G_F(x,x') = \theta(T-T')G^+(x,x') + \theta(T'-T)G^-(x,x')\,, \tag{21}$$

and the retarded and advanced Green's functions read

$$G_R(x,x') = \theta(T-T')\left(G^+(x,x') - G^-(x,x')\right)\,, \tag{22}$$

$$G_A(x,x') = \theta(T'-T)\left(G^+(x,x') - G^-(x,x')\right)\,, \tag{23}$$

where $G^+(x,x') - G^-(x,x')$ is the commutator function.

It is clear, though, that in the Schrödinger case, due to the fact that m is not summed over, there is no mixing between positive and negative frequency Wightman functions. For example, for $m > 0$ the Feynman propagator and the retarded Green's functions are the same, while for $m < 0$ the Feynman propagator equals the advanced Green's function.

The fact that in the Feynman propagator the step function $\theta(T-T')$ is multiplied by the step function $\theta(m)$ appearing in the Wightman function G^+ and similarly the fact that $\theta(T'-T)$ multiplies $\theta(-m)$ appearing in G^- has the following welcome consequence. Even though T is not a global time function and as such does not allow one to label all causally related events by a different value of T, it is not a problem to define a time ordering since the time ordering in the Feynman propagator is correlated with the sign of m. The failure of T to provide a well-defined global time ordering only applies to events with the same value of T. Propagation between such events with $m > 0$ or $m < 0$ does not occur.

By microcausality, the commutator function $G^+(x,x') - G^-(x,x')$ must vanish for spacelike separated points x and x'. In a free field theory the commutator function is a classical c-number quantity. Hence, it can only be nonzero whenever two points can be connected by a classical path. The commutator function is therefore zero for points that are either spacelike separated or that cannot be connected by a geodesic. Points separated in time such that $\sin\omega(T-T') = 0$ comprise the set of points that are either space-like separated ($T = T'$) or for which there is no geodesic connecting them. The $i\epsilon$ prescription is such that the commutator function vanishes for $\sin\omega(T-T')=0$. Hence the commutator function probes the following part of the space-time,

$$\bigcup_{n\in\mathbb{Z}} I^{+}(T=T'+(n-1)\tfrac{\pi}{\omega})\cap I^{-}(T=T'+n\tfrac{\pi}{\omega})\,, \tag{24}$$

which is the scalar field counterpart of the non-distinguishing character of space-time as seen by point particle probes.

4 Discussion

The way in which the scalar field theory resolves the point particle causal structure problems is via the unitary irreducible representations (6). The Hilbert spaces carry the momentum m in the direction of the lightlike lines as an additional label, a superselection parameter. This means that fields are always effectively massive (even when $m_0 = 0$). Hence in an eikonal approximation scalar fields correspond to particles moving along timelike curves and for massive point particles we did not observe any causal pathologies. For more information on this subject see [18]. It would be interesting to extend this analysis to include holographic renormalization (comparing with the work of [19]) and interacting field theories.

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