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SYMMETRIES AND PSEUDO-RIEMANNIAN MANIFOLDS

MATTHIAS BLAU*

Scuola Internazionale Superiore di Studi Avanzati, Trieste I-34014, Italy

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We generalize the classical Bochner–Yano theorems of Riemannian geometry to pseudo-Riemannian manifolds in order to obtain information on higher dimensional space-times with symmetries. The results are used to shed some light on questions of consistency and the zero-mode ansatz in Kaluza–Klein theories. Some applications to Einstein spaces are given.

Since our space-time is generally believed to be fairly well describable by a 4 (or $4+D$)-dimensional manifold of Lorentzian signature, it is of interest to know which of the powerful tools and theorems of Riemannian geometry (see e.g. [1]) may be carried over to the Lorentzian (or – more generally – to the pseudo-Riemannian) case.

Only fairly recently however has pseudo–Riemannian geometry attracted attention as a separate branch of mathematics (for a text-book account and references to earlier work see [2]). The reason for this may be that pseudo–Riemannian geometry is slightly disappointing from a mathematical point of view, since the theorems that can be proven are usually significantly weaker than their Riemannian counterparts. A notable exception of this are some "time-like" versions [3] of classical comparison-theorems [4]. (Here one essentially works inside the "Riemannian" light-cone.)

However, from a physical point of view one might – quite apart from the interest in its own right – be tempted to make a virtue out of this fact by saying that the pseudo–Riemannian case in less restrictive. One of the most striking features of Riemannian geometry is the intimate relationship and subtle interplay between curvature and topology (see [5], or [6] for some recent results) and curvature and symmetries (i.e. curvature and Killing vectors [5]).

Thus – since symmetries play an important role in classical general relativity

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* On leave of absence from the Institute of Theoretical Physics, University of Vienna, Austria.

and in order to obtain general information on the geometry of Kaluza–Klein theories [7] without going into the details of a specific model – it seems worth investigating what *a priori* constraints symmetries impose on the curvature and what conclusions may be drawn from the Einstein (or some other) field equations determining the geometry (locally) as regards the existence of symmetries.

The plan of this paper is therefore as follows:

In Section I we give a short summary of the original Bochner–Yano theorems [5] and indicate some of their applications. In Section II we illustrate the difficulties one encounters when trying to extend these results to pseudo-Riemannian manifolds and derive the analogue of the fundamental Bochner–Yano formula in this case. In Section III we give some applications of these results to Kaluza–Klein theories (relating the occurrence of the "Kaluza–Klein constraints" [8] to the setting sketched above and examining the traditional "zero-mode-ansatz" from this point of view), and to Einstein spaces.

I. The classical Bochner–Yano theorems

Let $(M, \langle , \rangle)$ be an n $(= D+4)$-dimensional compact orientable Riemannian manifold with metric $\langle , \rangle$ and denote by ∇ its unique metric and torsionfree (Levi-Civita) connection.

The norm of a vector field X (or tensor field T) will occasionally be denoted by $\|X\|$ (or $\|T\|$) instead of $\langle X, X\rangle = X_A X^A$ (or $\langle T, T\rangle = T_{AB\ldots C} T^{AB\ldots C}$) (upper case Latin indices run from 1 to $n = \dim M$). By straighforward computation it may be established that every vector field on M satisfies

$$\operatorname{div}[\nabla_X X - (\operatorname{div} X) X] = \operatorname{Ric}(X, X) + \operatorname{tr}(\nabla X)^2 - (\operatorname{div} X)^2, \tag{I.1}$$

where Ric is the Ricci tensor (with components R_{AB}), div is the divergence operator ($\operatorname{div} X = \nabla_A X^A$) and tr denotes the trace. Integrating this expression over M one obtains

$$\int_M [\operatorname{Ric}(X, X) + \operatorname{tr}(\nabla X)^2 - (\operatorname{div} X)^2] = 0 \tag{I.2}$$

and inserting the condition for X to be a Killing vector field

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0, \qquad \forall Y, Z \tag{I.3}$$

or a harmonic vector field

$$\begin{aligned} \langle \nabla_Y X, Z\rangle - \langle \nabla_Z X, Y\rangle = 0, \qquad \forall Y, Z, \\ \operatorname{div} X = 0 \end{aligned} \tag{I.4}$$

one arrives at

$$\int_M [\mathrm{Ric}(X, X) + \|\nabla X\|^2] = 0 \tag{I.5}$$

or

$$\int_M [\mathrm{Ric}(X, X) - \|\nabla X\|^2] = 0. \tag{I.6}$$

This allows one to conclude that there are no Killing vector fields on a compact orientable Riemannian manifold if $\mathrm{Ric} < 0$, and no harmonic vector fields if $\mathrm{Ric} > 0$. Thus the first Betti-number $b_1(M)$ of a compact orientable Riemannian manifold with $\mathrm{Ric} > 0$ (e.g. the n-sphere S^n for $n > 1$) is zero, and every closed one-form is exact. Analogous results for Killing and harmonic tensor fields may also be proved along these lines, the quadratic form replacing Ric is however somewhat more complicated.

A similar way of reasoning is employed in the proof of the Hopf–Bochner theorem [5], which states that if $\Delta f \geqslant 0$ for a function f on a compact Riemannian manifold, then f is constant

$$\Delta f \geqslant 0$$

and

$$\int_M \Delta f = 0$$

imply

$$\Delta f = 0.$$

Integrating Δf^2 over M leads to

$$\int_M \|\mathrm{grad}\, f\|^2 = 0,$$

which yields

$$f = \mathrm{const}.$$

The consequences of (I.5) are of relevance for Kaluza–Klein theories, since they tell us that we cannot simply use Ricci-flat internal spaces to circumvent the cosmological-constant-problem [7], if we want other than Abelian symmetries to emerge from this space.

II. The pseudo-Riemannian case

Let now M be a pseudo-Riemannian manifold, which we shall not assume to be compact for the time being, since compact Lorentzian manifolds are known to violate causality in the sense that they contain closed time-like curves [9]. (It is not clear, however, what role this "classical" causality violation (if any) plays at the Planck scale, which is the relevant regime for Kaluza–Klein theories.)

(I.1) remains true in this context and we could arrive at (I.2) by imposing suitable boundary conditions on X at infinity, but let us first investigate which conclusions may be drawn from (I.1) in the more general case.

In the case of geodesic Killing vector fields (which appear in the standard Kaluza–Klein ansätze [7]) or harmonic vector fields of constant length (which are of interest because of the Goldberg–Kobayashi theorem [5] and related results) we can drop the term on the left-hand side of (I.1) right away to obtain

$$\mathrm{Ric}(X, X)+\mathrm{tr}(\nabla X)^2 = 0. \tag{II.1}$$

This leads to

$$\mathrm{Ric}(X, X) = \|\nabla X\|^2 \tag{II.2}$$

for geodesic Killing vectors and

$$\mathrm{Ric}(X, X)+\|\nabla X\|^2 = 0 \tag{II.3}$$

for harmonic vectors of constant length and allows us to draw conclusions similar to those obtained from (I.5) and (I.6).

PROPOSITION A. *Let M be a Riemannian manifold (not necessarily compact). Then there are no geodesic Killing vector fields if* $\mathrm{Ric} < 0$ *and no harmonic vectors of constant length if* $\mathrm{Ric} > 0$.

The following proposition is relevant for Kaluza–Klein theories (see [8] and Section III (c)):

PROPOSITION B. *Let M be a Ricci-flat pseudo-Riemannian manifold. Then every geodesic Killing vector field and every harmonic vector field of constant length has to satisfy*

$$\|\nabla X\|^2 = 0. \tag{II.4}$$

If we impose such boundary conditions on the admissible vector fields that the left-hand side of (I.1) vanishes upon integration over M, we can do away with the assumptions of geodesy and constant length and arrive at (I.2) in this case as well.

Thus for Killing vector fields on Ricci-flat manifolds we obtain the condition – weaker than (II.4) –

$$\int_M \|\nabla X\|^2 = 0. \tag{II.5}$$

This is of interest because the restriction imposed on X in the Riemannian case (namely to be parallel and hence to generate only Abelian symmetries) can be avoided here and non-Abelian gauge symmetries may be obtained from Ricci-flat internal spaces (pseudo-Riemannian internal spaces have been suggested in [10] for entirely different reasons).

Several other results along these lines may be obtained by exploiting (I.1) in various situations. We shall, however, not pursue this matter here but rather just mention some results in the case of Einstein manifolds in the next chapter.

As an illustration of the difficulties one encounters in the pseudo-Riemannian case, let us try to derive an analogue of the Hopf–Bochner theorem. Since the Laplacian is now hyperbolic rather than elliptic, it is to be expected that the result $f = \text{const}$ does not persist in this case. It seems to be less generally well known however, what conclusions may nevertheless be drawn from $\Delta f \geqslant 0$. Thus assume $\Delta f \geqslant 0$ and (for simplicity) that f goes to zero at infinity sufficiently fast. Then we arrive – imitating the proof of Chapter I – at the conclusion that f necessarily has to satisfy $\Delta f = 0$ and

$$\int_M \|\text{grad}\, f\|^2 = 0. \tag{II.6}$$

A sufficient condition for (II.6) to be satisfied is that grad f be null – $\|\text{grad}\, f\|^2 = 0$. However, not even this condition – already weaker than its Riemannian counterpart – is necessary, as the following example shows:

Choose $M = T^2 = S^1 \times S^1$ (with coordinates t and x) and equip it with the standard metric but non-standard signature $(-\,+)$. Then every solution of $\Delta f = 0$ is of the form

$$f(t, x) = f_1(t+x) + f_2(t-x) \tag{II.7}$$

and

$$\|\text{grad}\, f\|^2 = -4\dot{f}_1\,\dot{f}_2 \neq 0$$

(a $\dot{}$ denotes derivative with respect to t), however

$$\int_{T^2} \|\text{grad}\, f\|^2 = 0.$$

III. Applications

Let us now turn to some applications of these ideas. Certain results – arising in the context of Kaluza–Klein theories – have already been mentioned above:

(a) If the internal space is compact and Riemannian, its Ricci-tensor has to be positive definite in order to allow for the occurrence of non-Abelian symmetries.

(b) If the internal space is pseudo-Riemannian, then Killing vector fields satisfying $\int \|\nabla X\|^2 = 0$ may lead to non-Abelian symmetries in the Ricci-flat case.

As an application of Proposition B we mention [8]:

(c) Let $M = M_4 \times G$ be the product of a pseudo-Riemannian 4-manifold M_4 and a compact connected semi-simple Lie-group G. Use the standard Kaluza–Klein ansatz [7] for a metric on M; then – due to the presence of Killing vectors of the total metric – the G-Yang–Mills field configurations are constrained to satisfy

$$F_{amn} F_b{}^{mn} = -g_{ab} \tag{III.1}$$

(F_a is the Yang–Mills field-strength, $-g_{ab}$ the Killing–Cartan form of G, and m, n label world-indices on M_4).

In the case $M = M_4 \times S^1$ Proposition B analogously leads to the result

$$F_{mn} F^{mn} = 0 \tag{III.2}$$

for the electromagnetic field-strength F_{mn}. These constraints have been discussed by Duff *et al.* [11] from a different point of view in their "Kaluza–Klein consistency" programme.

(d) An interesting aspect of the Bochner–Yano operator of the internal space

$$\nabla^2 + \mathrm{Ric}$$

is the fact [1] that it appears as the vector-mass-operator, when small fluctuations of the metric around its ground-state value are taken into account. This gives a partial justification of the zero mode ansatz [7] of Kaluza–Klein theories, since every Killing vector field of the internal space satisfies

$$\nabla^2 X^A + R^A_B X^B = 0. \tag{III.3}$$

However, not all solutions of (III.3) are Killing vector fields. But the additional solutions (like conformal Killing vectors in the case of two internal dimensions – giving rise to massless conformal vector-modes) are usually eliminated by imposing $\operatorname{div} X = 0$ as a gauge condition. Since these gauge conditions are to a certain extent arbitrary, the physical interpretation of these modes is, however, not quite clear.

(e) As a final application in the case of Einstein spaces

$$\mathrm{Ric} = c\,\langle\, , \rangle, \qquad c \in \boldsymbol{R} \tag{III.4}$$

let us mention the following facts which are easy corollaries of the general formulae (I.1) and (II.1)–(II.3):

COROLLARY A. *Let M be a pseudo-Riemannian Einstein space* (III.4). *If M admits a parallel vector field, then either $c = 0$ or the vector field is null.*

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[1] I am grateful to Jan Sobczyk for drawing my attention to this.

COROLLARY B. *On a pseudo-Riemannian Einstein space M every null geodesic field (world lines of massless particles) with tangent vector field X satisfies*

$$\operatorname{tr}(\nabla X)^2 + X \operatorname{div} X = 0.$$

COROLLARY C. *Let M be a compact orientable Ricci-flat Riemannian manifold. Then*

$$\int_M \operatorname{tr}(\nabla X)^2 = \int_M (\operatorname{div} X)^2, \qquad \forall X.$$

IV. Conclusions

We have generalized some classical theorems of Riemannian geometry due mainly to Bochner and Yano to pseudo-Riemannian manifolds, in order to obtain some general information on space-times with symmetries. Due to the fact that the differential operators involved are hyperbolic rather than elliptic in that case and because the space-time topology is not the (non-Hausdorff) metric topology, the results were somewhat weaker than in the Riemannian case. Nevertheless it would be of interest to see, what other Lorentzian analogues of "Riemannian" results will find application in physics and what may be learnt from these.

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REFERENCES

[1] Bishop R. L. and R. J. Crittenden: *Geometry of Manifolds*, Academic Press, New York, 1964.
[2] Beem J. K. and P. E. Ehrlich: *Global Lorentzian Geometry*, M. Dekker, New York, 1981.
O'Neill B.: *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
[3] Harris S. G.: *Ind. Univ. Math. J.* **31** (1982), 289.
[4] Cheeger J. and D. G. Ebin: *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.
[5] Yano K. and S. Bochner: *Curvature and Betti Numbers*, Annals of Mathematical Studies No. 32, Princeton University Press, Princeton, 1953.
Goldberg S. I.: *Curvature and Homology*, Academic Press, New York, 1962.
[6] Shiohama K. *et al.* (eds.): *Curvature and Topology of Riemannian Manifolds*, Lecture Notes in Mathematics 1201, Springer, New York, 1986.
[7] Kaluza Th.: *Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys.* (1921), 966.
Klein O.: *Z. Phys.* **37** (1926), 895.
Trautmann A.: *Rep. Math. Phys.* **1** (1970), 29.
Cho Y. M.: *J. Math. Phys.* **16** (1975), 463.
Duff M. J., B. E. W. Nilsson and C. N. Pope: *Phys. Rep.* **130** (1986), 1.

[8] Blau M., *Killing Vectors, Constraints and Conserved Charges in Kaluza–Klein Theories*, SISSA preprint 82/86/E.P. (to appear in *Class. Quant. Grav.*).
Blau M., G. Landi and W. Thirring: *An Introduction to Kaluza–Klein Theories*, University of Vienna Preprint UWTHPH-1986-5 and *Proceedings of the Schladming Winterschool 1986* (Springer, 1987, to appear).
[9] Hawking S. W. and G. F. R. Ellis: *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973, p. 189.
[10] Arafeva I. Ya. and I. V. Volovich: *Phys. Lett.* **164B** (1985), 287.
[11] Duff M. J., B. E. W. Nilsson, C. N. Pope and N. Warner: *Phys. Lett.* **149B** (1984), 90.
Duff M. J., Plenary talk, 4th Marcel Grossmann Meeting on General Relativity, Rome 1985.