Letters in Mathematical Physics **13** (1987) 83–92.

Connections on Clifford Bundles and the Dirac Operator

M. BLAU
Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria

(Received: 10 October 1986)

Abstract. It is shown, how, in the setting of Clifford bundles, the spin connection (or Dirac operator) may be obtained by averaging the Levi–Civita connection (or Kähler–Dirac operator) over the finite group generated by an orthonormal frame of the base manifold.

The familiar covariance of the Dirac equation under a simultaneous transformation of spinors and matrix representations emerges very naturally in this scheme, which can also be applied when the manifold does not possess a spin structure.

1. Introduction

In recent years, there has been a growing interest in a nonstandard description of fermions by differential forms instead of spinors.

Originally proposed by Ivanenko and Landau as early as 1928 [1], this approach was independently discovered by Kähler in 1960 [2], who suggested the equation

$$(d + \delta)\psi = m\psi . \tag{1.1}$$

Here ψ is an inhomogeneous differential form on Minkowski space, d is the exterior derivative, and δ the exterior coderivative $\delta = *^{-1} d*$, where $*$ is the Hodge duality operator [12]. He showed that this equation is completely equivalent to the Dirac equation (even if minimal electromagnetic coupling is introduced), provided that ψ lies in a minimal left ideal of the exterior algebra.

To see the relation to the standard description, note that $d + \delta$ is also a 'square-root' of the Laplace operator, since

$$(d + \delta)^2 = d\delta + \delta d = \Box \tag{1.2}$$

and that the restriction on ψ reduces the number of components from 16 to 4.

These ideas were generalized to manifolds by Graf [3], who was the first to point out that in a curved spacetime the Kähler–Dirac equation (1.1) and the Dirac equation are not necessarily equivalent.

Subsequent work by Benn and Tucker [4–5] has brought about many improvements and refinements of the original theory which is by now a fairly well developed alternative to the usual fermion = spinor philosophy.

Particular attention has been paid to these 'Kähler fermions' in the context of the lattice theories [17], where they seem to offer a promising way to circumvent some difficulties of the standard approach.

It is the purpose of this Letter to establish a relation between the Dirac equation and the Kähler–Dirac equation. The necessary mathematical framework is introduced in Section 2. Section 3 deals with the Levi-Civita connection on Clifford bundles and, in Section 4, the techniques are developed which allow us to obtain the Dirac equation from the Kähler–Dirac equation by a simple averaging procedure, as is shown in Section 5. There it is also pointed out how the usual copled transformation of spinors and γ-matrices arises very naturally in this setting. Finally some concluding remarks may be found in Section 6.

2. Mathematical Prelminaries

2.1. MULTILINEAR ALGEBRA

Let E be a finite (n)-dimensional real vector space equipped with a nondegenerate quadratic form Q and its associated bilinear form g

$$g(e,f) := \tfrac{1}{2}(Q(e+f) - Q(e) - Q(f)), \quad e, f \in E.$$

(Most of our considerations carry over to the degenerate case, but since g will later be taken to be the spacetime metric, we do not need this generality here.)

Denote by $\otimes E$ the tensor algebra of E, and by $I(E)$ (resp. $J(E)$) the ideals of $\otimes E$ generated by elements of the form $e \otimes e$ (resp. $e \otimes e - Q(e)$). Then the exterior algebra ΛE and the Clifford algebra $C(E)$ are defined by [7]

$$\Lambda E := \otimes E / I(E) \tag{2.1}$$

and

$$C(E) := \otimes E / J(E). \tag{2.2}$$

Multiplication in the resulting algebras will be denoted by '$\wedge$' (resp. '$\vee$') and their defining properties are

$$e \wedge F = -f \wedge e \tag{2.3}$$

and

$$e \vee f + f \vee e = 2g(e, f) \tag{2.4}$$

for $e, f \in E \subset \Lambda E$ (resp. $C(E)$).

The exterior algebra ΛE inherits the Z-graduation of $\otimes E$ (since the generators of $I(E)$ are of homoggneous Z-degree in $\otimes E$),

$$\Lambda E = \bigoplus_{p=0} \Lambda^p E, \tag{2.5}$$

and the elements of $\Lambda^p E$ are totally antisymmetric tensors of rank p.

Of considerable importance are the main automorphism η and the main anti-automorphism ξ defined by

$$\begin{aligned} \eta e &= -e, && e \in E \subset \otimes E, \\ \eta(e \otimes f) &= \eta e \otimes \eta f, && e, f \in \otimes E; \end{aligned} \tag{2.6}$$

$$\begin{aligned} \xi e &= e, && e \in E \subset \otimes E, \\ \xi(e \otimes f) &= \xi f \otimes \xi e, && e, f \in \otimes E, \end{aligned} \tag{2.7}$$

which survive the quotient-forming and act on ΛE and $C(E)$ as

$$\begin{aligned} \eta e &= (-)^p e, && e \in \Lambda^p E, \\ \eta(e \wedge f) &= \eta e \wedge \eta f, && e, f \in \Lambda E, \\ \eta(e \vee f) &= \eta e \vee \eta f, && e, f \in C(E); \end{aligned} \tag{2.8}$$

$$\begin{aligned} \xi e &= (-)^{\binom{p}{2}} e, && e \in \Lambda^p E, \\ \xi(e \wedge f) &= \xi f \wedge \xi e, && e, f \in \Lambda E, \\ \xi(e \vee f) &= \xi f \vee \xi e, && e, f \in C(E). \end{aligned} \tag{2.9}$$

η induces a Z_2-graduation on $\otimes E$,

$$\otimes E = \otimes^{(0)} E + \otimes^{(1)} E,$$

by

$$\otimes^{(i)} E = \{e \in \otimes E : \eta e = (-)^i e\}$$

which passes to ΛE and $C(E)$, because the generators $I(E)$ and $J(E)$ are even (i.e., elements of $\otimes^{(0)} E$).

A crucial observation is that one can define a Clifford product on ΛE by

$$e \vee f := e \wedge f + g(e, f), \quad e, f \in E \subset \Lambda E. \tag{2.10}$$

Indeed

$$e \vee f + f \vee e = 2g(e, f)$$

and the universal property of Clifford algebras [7] guarantees that the resulting algebra is isomorphic to $C(E)$.

Conversely an exterior (or wedge-) product may be defined on $C(E)$ by

$$\begin{aligned} e \wedge f &:= \tfrac{1}{2}(e \vee f - f \vee e) \\ &= e \vee f - g(e, f) \\ &= -f \wedge e. \end{aligned} \tag{2.11}$$

The relations (2.9) and (2.10) can be extended to p-vectors ψ by

$$e \vee \psi = e \wedge \psi + i_e \psi, \qquad \psi \vee e = \psi \wedge e - i_e \eta \psi \tag{2.12}$$

where i (an anti-derivation of degree -1) is the usual interior derivative defined by contraction with respect to g:

$$u_e f := g(e, f), \quad e, f \in E,$$
$$i_e(f_1 \wedge f_2) = (i_e f_1) \wedge f_2 + \eta f_1 \wedge i_e f_2, \tag{2.13}$$
$$i_e(f_1 \vee f_2) = (i_e f_1) \vee f_2 + \eta f_1 \vee i_e f_2.$$

The resulting algebra, in which '$\wedge$', '$\vee$' and 'i' are related in this way, is known as the Kähler–Atiyah algebra KA(E) [2, 3, 8, 9], and its elements, which may equivalently be regarded as antisymmetric inhomogeneous tensors or Clifford multi-vectors, will, as a rule, be denoted by Greek letters.

2.2. SPINORS

According to Brauer and Weyl [10], spinors are elements of irreducible Clifford modules, while the Wedderburn theorem on simple associative algebra tells us that these are isomorphic to minimal left ideals of the corresponding Clifford algebra [11].

Since minimal left ideals of Clifford algebras are themselves irreducible modules (the representation simply being left multiplication), spinors may be regarded as elements of minimal left ideals of Clifford algebras [9, 16].

Hence, in the setting sketched above, spinors may be equivalently regarded as certain antisymmetric tensors, and this will lead to the description of spinor fields in terms of differential forms below.

Given a minimal left ideal $S(E)$ of $C(E)$,

$$S(E) = C(E) \vee p, \tag{2.14}$$

where p is a primitive idempotent of $C(E)$, a necessary and sufficient condition for $\psi \in C(E)$ to be an element of $S(E)$ is

$$\psi \in S(E) \Leftrightarrow \psi = \psi \vee p. \tag{2.15}$$

Henceforth, objects ψ satisfying this condition will be called spinors (or, although the distinction is not necessary, algebraic spinors).

More information on the details of the relation among Clifford algebras, exterior algebras and spinors, and examples may be found in [9].

2.3. EXTENSION TO VECTOR BUNDLES

Almost everything we have said so far remains true if E is a smooth vector bundle equipped with a nondegenerate fibre metric instead of a vector space.

We then have the tensor bundle $\otimes E$, the ideal bundles $I(E)$ and $J(E)$, the quotient bundles (cf. (2.1), (2.2)) ΛE and $C(E)$ and the Kähler–Atiyah bundle KA(E). Given any bundle F over a smooth manifold M, the $C^\infty(M)$-module of smooth sections of F will be denoted by $\Gamma(F)$.

In order to make contact with the desired description of spinors on M in terms of differential forms, we choose $E = T^*M$ – the cotangent bundle of an orientable manifold M.

It should be noted that on a Lorentzian four-dimensional manifold with signature $(-\ +\ +\ +)$ (resp. $(+\ -\ -\ -)$) $C(T^*M)$ gives rise to four-component Majorana–Dirac (resp. two-component quaterionic) spinors. The usual Dirac spinors are, therefore, obtained by complexifying T^*M (or $C(T^*M)$: $C(T^*M \otimes C) \simeq C(T^*M) \otimes C$).

Let us choose an orthonormal frame of one-forms $\{e^\mu\}$ for T^*M. We then have (cf. (2.3), (2.4), (2.12))

$$e^\mu \wedge e^\nu = -e^\nu \wedge e^\mu ,$$
$$e^\mu \vee e^\nu + e^\nu \vee e^\mu = 2\eta^{\mu\nu} , \tag{2.16}$$

$$e^\mu \vee e^\nu = e^\mu \wedge e^\nu + \eta^{\mu\nu} ,$$
$$e^\mu \vee \psi = e^\mu \wedge \psi + i_{e^\mu}\psi =: e^\mu \wedge \psi + i^\mu\psi , \tag{2.17}$$

where $\eta^{\mu\nu} = g(e^\mu, e^\nu)$, g a metric on M.

Until now, no complications have arisen as a consequence of the transition from vector spaces to vector bundles, all the bundles introduced so far are well defined and exist globally. However, in order to relate spinor fields to certain sections of the Clifford bundle $C(T^*M)$, we need a globally defined minimal ideal S (2.14) of $C(T^*M)$, which may not always exist.

Since, to the best of the author's knowledge, the question of which manifolds do admit such an S has not been settled yet, we shall assume the existence of a globally defined primitive idempotent p (which is a sufficient but possibly not necessary condition) and can thus identify spinor fields with sections of the corresponding ideal bundle

$$\{\text{spinor fields}\} \Leftrightarrow \Gamma(C(T^*M) \vee p) = \Gamma(S(T^*M)) ,$$
$$\psi \in \Gamma(S(T^*M)) \Leftrightarrow \psi = \psi \vee p . \tag{2.15}$$

Regarding spinor fields as sections of $S(T^*M)$ has – while being completely equivalent to the standard point of view – many computational and conceptional advantages and allows a direct comparison of the Kähler–Dirac equation (1.1) with the Clifford algebraic formulation of the Dirac equation on S (5.3) due to Benn and Tucker [6] (cf. also [5]).

Because differential forms on M may be regarded as sections of $C(T^*M)$, we may, using (2.17), rewrite the Kähler–Dirac operator as

$$(d + \delta)\psi = e^\mu \wedge \nabla_\mu\psi + i^\mu\nabla_\mu\psi = e^\mu \vee \nabla_\mu\psi \tag{2.18}$$

where ∇ is the Levi–Civita connection of the (pseudo-) Riemannian manifold (M, g) and $\nabla\mu = \nabla_{e_\mu}$ is the covariant derivative along the vector field e_μ dual to e^μ.

Therefore, in order to relate the Dirac operator to the Kähler–Dirac operator, we shall first take a slightly more detailed look at the Levi–Civita connection on $C(T^*M)$.

3. The Levi–Civita Connection

The Levi–Civita connection of M is uniquely determined by the conditions [12]

$$de^\mu = -\omega^\mu_\nu \wedge e^\nu \quad \text{(torsion-free)},$$

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \quad \text{(Riemannian)},$$

on the connection-forms ω^μ_ν and acts on e^μ as

$$\nabla e^\mu = -\omega^\mu_\nu \otimes e^\nu \tag{3.1}$$

i.e.,

$$\nabla_X e^\mu = -(i_X \omega^\mu_\nu) e^\nu \tag{3.2}$$

for the covariant derivative ∇_X along the vector field $X \in \Gamma(TM)$.

∇ extends (by linearity and postulation of the Leibniz rule) in a unique way to $\otimes T^*M$ and passes to the quotient $C(T^*M)$ to yield a well-defined covariant derivative there, since ∇ preserves the ideal $J(T^*M)$ (2.2):

$$\begin{aligned}&\nabla_X(\phi\otimes\phi - g(\phi,\phi))\\ &\quad = \tfrac{1}{2}[(\nabla_X\phi + \phi)\otimes(\nabla_X\phi+\phi) - g(\nabla_X\phi+\phi, \nabla_X\phi+\phi)] -\\ &\qquad - \tfrac{1}{2}[(\nabla_X\phi - \phi)\otimes(\nabla_X\phi-\phi) - g(\nabla_X\phi-\phi, \nabla_X\phi-\phi)]\end{aligned}$$

for all $\phi \in \Gamma(T^*M)$, $X \in \Gamma(TM)$. This shows that

$$\nabla_X(\phi\otimes\phi - g(\phi,\phi)) \in \Gamma(J(T^*M)) \simeq J(\Gamma(T^*M))$$

and, therefore, ∇ is an algebra connection [13] on $C(T^*M)$:

$$\nabla_X(\phi \vee \psi) = (\nabla_X\phi)\vee\psi + \phi\vee\nabla_X\psi, \quad \forall\phi,\psi\in\Gamma(C(T^*M)).$$

Setting $g \equiv 0$ in the above computation shows that the same conclusion holds for ∇ on ΛT^*M.

The crucial observation is now, that despite all these nice properties, ∇ does not as a rule map a minimal left ideal S of $C(T^*M)$ into itself

$$\psi\in S \Rightarrow \psi = \psi\vee p, \qquad \nabla_X(\psi\vee p) = (\nabla_X\psi)\vee p + \psi\vee\nabla_X p.$$

$\nabla_X p$ is not necessarily an element of S, and the condition (2.15)

$$\nabla_X p = (\nabla_X p)\vee p \tag{3.3}$$

leads to [3]

$$P \vee \nabla_X P = 0. \tag{3.4}$$

Equation (3.4) and its integrability conditions impose severe restrictions on the Petrov-type of the manifold [14].

Therefore, the Kähler–Dirac operator $d + \delta = e^\mu \vee \nabla_\mu$ (2.18) cannot be identical with the Dirac operator nor with any other differential operator on S, although (2.18) obvious coincides with the Dirac operator in the flat space limit.

Thus what we want, in order to have an ideal-preserving connection, is a covariant derivative, which is, in a sense, invariant under right multiplication. In the next section we show how such a connection with the desired property may be obtained from ∇, and in Section 5 we show that the resulting differential operator is, indeed, the usual covariant derivative for spinors in curved space.

4. Averaging of ∇

In order to arrive at the desired invariance, we make use of the fact that the e^{μ} ($\mu = 1, \ldots, n$, a set of orthonormal one-forms) generate a finite group G of 2^{n+1} elements under Clifford multiplication, namely

$$G = \{\pm 1\}, \pm e^{\mu_1} \vee e^{\mu_2} \vee \ldots \vee e^{\mu_p}: 1 \leqslant p \leqslant n, \mu_1 < \mu_2 < \cdots < \mu_p\}$$
$$=: \{e_{\pm I}, I = 1, \ldots, 2^n, e_{-I} := -e_I\}.$$

In the literature, these groups run under the name of 'vee-groups' [15], and the study of their properties gives insight into the structure and systematics of Clifford algebras [9, 15].

Now there is a general procedure to obtain a connection invariant under the action of some compact Lie group from a given connection. This procedure amounts to integrating (or averaging) the connection over the group [13].

In general, this may be quite cumbersome but, in our case (where we have a finite group), this integration reduces to a simple summation (or equivalently: the invariant volume element collapses to a discrete measure).

We shall specify the action of G to be fibre-wise left or right multiplication on 'its' Clifford algebra and (for computational ease, since ∇ acts from the left) compute the left invariant connection D (4.1) resulting from ∇ first. The right invariant connection $\bar{D}$ may be obtained from D by transforming it with the main anti-automorphism (2.9) in a suitable way (5.1).

Let us, therefore, compute

$$D = 2^{-(n+1)} \sum_{\pm I} e_I \vee \nabla_0 e_I^{-1} \vee$$
$$= 2^{-n} \sum_I e_I \vee \nabla_0 e_I^{-1} \vee$$
$$= \nabla + 2^{-n} \sum_I e_I \vee (\nabla e_I^{-1}) \vee$$
$$= \nabla - \tfrac{1}{2}\omega_{\mu\nu} \otimes \sigma^{\mu\nu} \vee \tag{4.1}$$
$$=: \nabla - \Sigma \vee \tag{4.2}$$

where

$$\sigma^{\mu\nu} := \tfrac{1}{4}(e^{\mu} \vee e^{\nu} - e^{\nu} \vee e^{\mu}).$$

Equation (4.1) is obtained by repeated application of (3.1), and looks quite promising, since we have (somewhat surprisingly?) produced the generators $\sigma^{\mu\nu}$ of the spin group [11] of the Clifford algebra by this averaging.

We may check by explicit calculation that D has indeed the desired property, namely

$$D(e^\mu \vee \psi) = e^\mu \vee D\psi\,, \quad \forall \psi \in \Gamma(C(T^*M))\,, \tag{4.3}$$

$$\begin{aligned} D(e^\mu \vee \psi) &= (\nabla e^\mu) \vee \psi + e^\mu \vee \nabla\psi - \Sigma \vee e^\mu \vee \psi \\ &= -\omega^\mu_\nu \otimes e^\nu \vee \psi + e^\mu \vee \nabla\psi - \\ &\quad - \tfrac{1}{2}\omega_{\nu\rho} \otimes e^\mu \vee \sigma^{\nu\rho}\psi - \tfrac{1}{2}\omega_{\nu\rho} \otimes (\eta^{\mu\rho}e^\nu - \eta^{\mu\nu}e^\rho)\psi \\ &= e^\mu \vee D\psi\,. \end{aligned}$$

In the following we list some basic properties of D:

(1) D is compatible with the '+'-operation on $C(T^*M)$.
(2) D is not an algebra connection on $C(T^*M)$:

$$D(e^\mu \vee \psi) = e^\mu \vee D\psi \neq (De^\mu) \vee \psi + e^\mu \vee D\psi\,.$$

(3) D is compatible with the Z_2-graduation of $C(T^*M)$, i.e., D maps even to even and odd to odd elements and, therefore, commutes with the main automorphism η (2.8), $D\eta = \eta D$.
(4) D does not commute with the main anti-automorphism ξ (2.9).
(5) D does not commute with right-'$\vee$'-multiplication: To evaluate $D(\psi \vee e^\mu)$, we shall regard $\psi \vee e^\mu$ as an element of $\mathrm{KA}(T^*M)$ and make use of (2.12):

$$\begin{aligned} D(\psi \vee e^\mu) &= D(\psi \wedge e^\mu - i^\mu \eta\psi) \\ &= D(e^\mu \wedge \eta\psi - i^\mu\eta\psi) \\ &= D(e^\mu \vee \eta\psi - 2i^\mu\eta\psi) \\ &= e^\mu \vee \eta D\psi - 2Di^\mu\eta\psi) \\ &= e^\mu \wedge \eta D\psi + i^\mu\eta D\psi - 2Di^\mu\eta\psi \\ &= D\psi \wedge e^\mu + i^\mu\eta D\psi - 2Di^\mu\eta\psi \\ &= D\psi \vee e^\mu + 2i^\mu\eta D\psi - 2Di^\mu\eta\psi \\ &= (D\psi) \vee e^\mu + 2\eta[D, i^\mu]\psi\,. \end{aligned} \tag{4.4}$$

Since $[D, i^\mu]$ does not vanish identically, the left invariance of D does not imply its right invariance.

5. The Dirac Equation

Let us define $\bar{D}$ by

$$\bar{D}\psi := \xi D \xi \psi\,. \tag{5.1}$$

Then we find

$$\begin{aligned}\bar{D}(\psi \vee e^{\mu}) &= \xi D\xi(\psi \vee e^{\mu}) \\ &= \xi D(\xi e^{\mu} \vee \xi\psi) \\ &= \xi D(e^{\mu} \vee \xi\psi) \\ &= \xi(e^{\mu} \vee D\xi\psi) \\ &= \xi D\xi\psi \vee e^{\mu} \\ &= (\bar{D}\psi) \vee e^{\mu}. \end{aligned} \tag{5.2}$$

Thus, $\bar{D}$ is right invariant and coincides with the connection we would have obtained by averaging ∇ over the right action of G.

In terms of connection forms, $\bar{D}$ is given by

$$\begin{aligned}\bar{D}\psi = \xi D\xi\psi &= \xi\nabla\xi\psi - \tfrac{1}{2}\omega_{\mu\nu} \otimes \xi(\sigma^{\mu\nu} \vee \xi\psi) \\ &= \nabla\psi + \tfrac{1}{2}\omega_{\mu\nu} \otimes \psi \vee \sigma^{\mu\nu} \\ &= \nabla\psi + \psi \vee \Sigma . \end{aligned} \tag{5.3}$$

As shown by Benn and Tucker [6], this operator $\bar{D}$ yields precisely the curved space Dirac equation if one chooses a primitive idempotent p constructed from the $\{e_I\}$ and the corresponding matrix-basis.

Since then $\nabla p = [\Sigma, p]$ is a consequence of $\nabla e^{\mu} = [\Sigma, e^{\mu}]$, we obtain

$$\begin{aligned}\psi = \psi \vee p \Rightarrow \bar{D}\psi = \bar{D}(\psi \vee p) &\\ &= (\nabla\psi) \vee p + \psi \vee \nabla p + \psi \vee p \vee \Sigma \\ &= (\nabla\psi) \vee p + \psi \vee [\Sigma, p] + \psi \vee p \vee \Sigma \\ &= (\nabla\psi + \psi \vee \Sigma) \vee p \\ &= (\bar{D}\psi) \vee p . \end{aligned} \tag{5.4}$$

If one averages ∇ not with respect to $\{e_I\}$ but, say, with respect to the 'conjugate' group $\{S^{-1} \vee e_I \vee S\}$, corresponding to a change of the γ-matrix representation, thereby obtaining $\bar{D}[S^{-1} \vee e_I \vee S]$ instead of $\bar{D}[e_I]$, one sees that

$$\begin{aligned}&\bar{D}[S^{-1} \vee e_I \vee S](\psi \vee S) \\ &\quad = \bar{D}[S^{-1} \vee e_I \vee S](\psi \vee S \vee S^{-1} \vee p \vee S) \\ &\qquad = (\bar{D}[S^{-1} \vee e_I \vee S](\psi \vee S)) \vee S^{-1} \vee p \vee S .\end{aligned}$$

Therefore, one recovers the familiar covariance

$$\psi \to \psi \vee S , \tag{5.5}$$

$$e_I \to S^{-1} \vee e_I \vee S \tag{5.6}$$

of the Dirac equation very elegantly in this setting and one can interpret it as adjusting the representation in such a way that the e_I have constant components with respect to the corresponding matrix basis. (In (5.5), S only seems to act on ψ from the wrong side: upon spelling out (5.5) in a matrix basis one recovers the usual left action of S on the components of ψ.)

6. Conclusion

We have shown that, starting with the Kähler–Dirac operator (1.1), (2.15)), $e^{\mu} \vee \nabla_{\mu}\psi = (d + \delta)\psi$ we may arrive at the Dirac operator (5.3)

$$e^{\mu} \vee \bar{D}_{\mu}\psi = e^{\mu} \vee \nabla_{\mu}\psi + e^{\mu} \vee \psi \vee i_{\mu}\Sigma$$

by averaging the Levi–Civita connection ∇ over the finite group generated by the e^{μ}, thereby ensuring the desired ideal preserving property of $\bar{D}$.

While being interesting in itself, this new geometrical interpretation of the Dirac operator gives some insight into the interplay between the global conditions under which the procedure outlined above works and the question of existence of a spin-structure on M, which is quite subtle and presently under investigation.

Acknowledgements

I want to thank Ian Benn for some enlightening remarks on these topics, and Helmuth Urbantke for numerous helpful discussions and a critical reading of the manuscript.

References

1. Ivanenko, D. and Landau, L., *Z. Phys.* **48**, 340 (1928).
2. Kähler, E., *Abh. Deutsch. Akad. Wiss. Berlin* (*Math.-Phys.*) 4 (1960) *and* **1** (1961); *Rend. Mat.* **21**, 425 (1962).
3. Graf, W., *Ann. Inst. H. Poincaré* **A29**, 85 (1978).
4. Benn, I. M. and Tucker, R. W., *Commun. Math. Phys.* **89**, 34 (1983); *Nuovo Cim.* **88A**, 273 (1985).
5. Benn, I. M. and Tucker, R. W., *Phys. Lett.* **130B**, 177 (1983).
6. Benn, I. M. and Tucker, R. W., *Commun. Math. Phys.* **98**, 53 (1985).
7. Greub, W., *Multilinear Algebra*, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin, 1978.
8. Atiyah, M. F., *Arbeitsgem. Forschung Landes Nordrhein-Westfalen, Naturwissenschaften* **200** (1970).
9. Blau, M., 'Clifford Algebras and Kähler–Dirac Spinors', University of Vienna preprint, UWThPh-1986-16, and Master's Thesis (unpublished).
10. Brauer, R. and Weyl, H., *Am. J. Math.* **57**, 425 (1935).
11. Chevalley, C., *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.
12. Thirring, W., *A Course on Mathematical Physics, Vol. 2, Field Theory*, 2nd ed., Springer, New York, 1986.
13. Greub, W., Halperin, S., and Vanstone, R., *Connections, Curvature and Cohomology*, Vol. II, Academic Press, New York, London, 1973.
14. Al Saad, A. and Benn, I. M., 'Ideal Preserving Lorentzian Connections', University of Lancaster preprint (1985).
15. Salingaros, N., *J. Math. Phys.* **22**, 226 (1981), and **23**, 1 (1982).
16. Riesz, M., *Clifford Numbers and Spinors*, Lecture Series No. 38, University of Maryland, College Par, 1958.
17. Becher, P. and Joos, H., *Z. Phys.* **C15**, 343 (1982).