ANNALS OF PHYSICS 205, 130–172 (1991)

Topological Gauge Theories of Antisymmetric Tensor Fields

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Received April 23, 1990

We introduce a new class of topological gauge field theories in any dimension, based on anti-symmetric tensor fields, and discuss the BRST-quantization of these reducible systems as well as the equivalence of BRST-quantization and Schwarz's method of resolvents in detail. As a consequence we can use path-integral techniques and BRST-symmetry to prove metric independence and other properties of the Ray–Singer torsion. We pay particular attention to the presence of zero modes and discuss various methods of treating them in these models and other topological field theories. Non-Abelian models in two dimensions provide us with a complete Nicolai map for Yang–Mills theory on an arbitrary two-surface, as well as with a theory of topological gravity which is closely related to Hitchin's self-duality equation on a Riemann surface. Candidate observables for Abelian models in any dimension are linking and intersection numbers of manifolds for which we give explicit path-integral representations.

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0003-4916/91 $7.50

1. INTRODUCTION

A novel class of field theories was introduced recently by Witten [1–3], which are purely topological in the sense that their partition functions are independent of the metric and that the only observables in these theories are topological (in the sense of smooth) invariants of the underlying space-time manifold M.

These topological field theories (TQFTs) can basically arise in two different ways. On the one hand, the quantum action S_q appearing in the path-integral treatment of these theories may be obtained by the BRST gauge fixing of a huge (topological) symmetry which permits all physical fields to be gauged away locally, thus leaving only one with the (usually finite-dimensional!) space of zero-modes as the true configuration space of the theory. The underlying classical action may either be zero [4], or have a form [5–7] which makes it obvious, that all physical (propagating) fields may be eliminated by a shift of integration variables in the path-integral. The latter formulation makes it particularly easy to see [6], that this type of TQFT has the aesthetically appealing feature of possessing a complete (non-perturbative) Nicolai map [8, 9], which trivializes the path-integral over all but a finite-dimensional subspace (moduli-space) of the space of fields. The models of [1, 2] are of this type, and Witten has shown how—following the influential suggestions of Atiyah [10]—they may be regarded as field-theoretic descriptions of certain remarkable developments in the study of two-[11], three-[12], and four-manifolds [13].

On the other hand, theories based on the Chern–Simons action in three dimensions have also been shown by Witten [3] to lead to TQFTs in the sense above. On the mathematical side, these are deeply related to the intrinsically three-dimensional knot-theory and its Jones polynomials [14], whereas physically they provide a three-dimensional understanding of a large class of conformal field theories in two dimensions and an honest gauge theory interpretation of three-dimensional Einstein gravity [15].

Due to the fascinating phenomena both these types of TQFTs exhibit, it is certainly of interest to look for other models in other dimensions, which also qualify as topological field theories. But while the former type of theories clearly has great flexibility and exists in many dimensions, the Chern–Simons functional is firmly rooted in three, and the higher-dimensional Chern–Simons terms—although interesting in their own right—are somewhat problematical from the physical point of view, since they contain higher order derivatives.

However a far-reaching suitable generalization of the Chern–Simons theory does exist and suggests itself by realizing that the relevant property of the Chern–Simons theory is the *metric independence of the classical action*, a property shared by any classical action written in terms of differential forms, without reference to the Hodge duality operator.

Indeed a large class of TQFTs, i.e., field theories whose partition functions are independent of the metric on an arbitrary manifold, may be obtained from these classical actions. The reason for this is that for a large subset of these theories the BRST-gauge fixed quantum action will differ from the classical action only by a BRST-commutator which contains the whole metric dependence (gauge-fixing terms, ghost kinetic term) of the quantum action. This in turn implies that the metric variation of the partition function will be the vacuum expectation value of a BRST-commutator and hence zero if the vacuum is BRST-invariant.

In more general theories (usually those plagued by on-shell reducible symmetries) the above argument is not directly applicable, since unlike the Faddev–Popov and BRST-procedures the more general Batalin–Vilkovisky algorithm [16] which is required here does not guarantee that the quantum action is just the classical action plus a BRST-commutator, But—as we shall see—we can go a long way towards proving metric independence in those theories which allow for a usual path-integral treatment. This restriction implies that we consider only those actions, whose quadratic parts are first order in derivatives (since the usual second-order kinetic terms are ruled out by the requirement of metric independence). This feature—as well as the result (Section 2.4) that these theories have no physical degrees of freedom—is shared by the Chern–Simons action.

Such actions will be called (rather loosely) BF systems (the name coming from our later discussion of non-Abelian theories, where B will be an $(n-2)$-form and $F=F_A$ is the curvature two-form of some connection A, though we do not restrict ourselves to just these models), and in this paper we shall explore some of the interesting features these BF systems display.

Returning now first to the question of metric independence of the partition function, we shall see that for all Abelian BF systems in arbitrary dimensions (Section 2.3) as well as their non-Abelian counterparts in two and three dimensions (Section 3) the argument sketched above (i.e., as a consequence of the fact that the gauge-fixed action may be written as $S+\int \{Q, \Psi\}$) is sufficient to establish this. For the non-Abelian models in higher dimensions we present various pieces of circumstantial evidence in support of our conjecture that they also lead to TQFTs. For instance, when one specializes to n-manifolds of the form $M_n=\Sigma_{n-1}\times R$ an alternative proof is available, as is for arbitrary four-manifolds (Section 3.4).[§]

BF systems incorporate and generalize a class of models introduced by Schwarz [17], who used these actions to give a path-integral representation of the

[§] See note added in proof.

Ray–Singer torsion [18]. In that paper he anticipated much of the later developments on the quantization of theories with reducible symmetries (i.e., systems which require the ghost for ghost mechanism). Indeed the construction of Schwarz provides the proper geometrical framework for an understanding of the algebraic BRST-procedure. As BRST-invariance is the cornerstone of our proof that the quantum theories so obtained are also metric independent, we present a rather detailed comparison of this approach with that of Schwarz (Section 2).

As a consequence we shall be able to prove the metric independence of the Ray–Singer torsion as well as other properties (like its triviality in even dimensions) using path-integral techniques and BRST-symmetry (Section 2.5).

The classical action being linear in derivatives means that the usual rules employed to count degrees of freedom in the BRST-framework do not match those presented by Schwarz and are incorrect in this situation. A correct count (a la BRST) will be given reconciling the two (Section 2.4), and the new count of degrees of freedom is shown to be easily read off from the Batalin–Vilkovisky ghost-triangle.

Since in topological field theories a proper evaluation of expectation values requires a particularly careful handling of (harmonic) zero modes, we show in Section 2.6 how this can be done within the framework of Schwarz or via the Faddeev–Popov trick. The gauging of the harmonic modes is then shown to be equivalent to the more common prescription of inserting zero-modes into the measure of the path-integral. As an example we treat the chiral $b-c$ system on a Riemann surface of genus greater than one. We also present two other methods of handling zero-modes within the BRST-framework, which are particularly useful if one wants to keep the zero-mode integration explicit instead of gauging the zero-modes to zero. In Section 3.2 the latter approach will be used to take proper account of zero-modes in manipulations involving a Nicolai map.

To establish that our generalized models are not devoid of physical or mathematical interest, we consider in some detail (though not exhaustively) some examples. In particular we shall see that two-dimensional non-Abelian BF systems have some very nice properties. They allow us, for instance, to formulate a theory of topological gravity (living on the moduli space of Riemann surfaces), which is—as we shall explain—closely related to Hitchin's [19] deep investigations of the Yang–Mills self-duality equations on a Riemann surface (Section 3.3).

A second feature of BF systems in two dimensions is the existence of a Nicolai map (Section 3.2), which trivializes the path-integral, bringing these models closer to those of the type [1, 2]. This map is interestingly enough also a Nicolai map for two-dimensional Yang–Mills theory on an arbitrary two-surface and in arbitrary—also covariant—gauges, and in the particular case of the cylinder this permits us to recover the results of Rajeev [20], who recently solved this theory exactly.

Candidate observables for Abelian BF systems in higher dimensions are expectation values of Wilson loops and "Wilson surfaces," which can be exactly evaluated. They precisely describe intersection and linking numbers of manifolds, thus generalizing the path-integral representation of the linking number of two loops in three dimensions discovered by Polyakov [21] to arbitrary dimensions (Section 4).

We close by listing some open problems (Section 5), and in an appendix we sketch, how regularization of these theories can be performed in a BRST-invariant way.

2. Abelian Models

2.1. *General Aspects*

As an example of an Abelian BF system consider the following metric independent action on an n-dimensional manifold M (compact, without boundary),

$$S(n, p) = \int_M B_p \, dA_{n-p-1}, \tag{1}$$

where the fields A and B are forms (possibly taking values in some flat vector bundle), the subscript indicating their rank, d is the corresponding exterior derivative, and the wedge-product among forms will always be understood. The abelian gauge symmetries of this action,

$$\begin{aligned} \delta B_p &= dA_{p-1} \\ \delta A_{n-p-1} &= dA'_{n-p-2}, \end{aligned} \tag{2}$$

combined with the equations of motion following from (1),

$$\begin{aligned} dB_p &= 0 \\ dA_{n-p-1} &= 0, \end{aligned} \tag{3}$$

tell us that the space $\mathcal{N}$ of classical solutions (phase space) of the BF system (1) is a finite-dimensional vector space,

$$\mathcal{N} = H_d^p(M) \oplus H_d^{n-p-1}(M) \tag{4}$$

(where $H_d^k(M)$ is the kth deRham cohomology group of forms on M with values in a flat vector bundle). We can therefore regard the action (1) as providing us with a field theoretic description of the deRham complex on M. The quantization of (1) is straightforward and—as shown explicitly in Section 2.3—the quantum action is of the form

$$S_q(n, p) = S(n, p) + \int \{Q, \Psi\}, \tag{5}$$

where Q is the BRST-operator and Ψ is (in the parlance of [16]) the gauge fermion. Thus according to the argument of the previous section this action will give rise to a partition function which is a topological invariant. The question therefore arises: is this invariant non-trivial, and—if so—can it be identified with any known topological invariant?

Fortunately this question was already answered over 10 years ago by Schwarz [17] in a remarkable paper, which probably was the first to deal with the quantization of reducible theories. There he showed, that (for all values of p) the partition function of (5) is related to the Ray–Singer torsion [18], i.e., to the torsion of the deRham-complex of M. Since for $n>2$ the gauge symmetries (2) are reducible and Schwarz's method of resolvents is ideally suited to handle theories described by actions of the form (1), we shall discuss his method and its equivalence with BRST-quantization in some detail below. Before plunging into the technical details of this, however, we shall first give a heuristic discussion of reducibility.

2.2. *Resolvents and Quantization of Reducible Systems*

The picture one has in mind when using the Faddeev–Popov technique is that is provides one—despite the fact that one explicitly chooses a gauge—with a gauge-invariant way of factoring the volume of the whole gauge group out of the path-integral. This is what the Faddeev–Popov ghosts do.

It may happen, however, that not the whole would-be gauge group acts effectively on the space of fields. For instance, in the case of the action (1) one can quotient out directly the space of $(p-1)$-forms $\Omega^{p-1}(M)$. In doing so, however, one ignores the fact that, e.g., forms $\Lambda_{p-1}\in d\Omega^{p-2}(M)$ do not transform B at all in (2). In the BRST-framework this is reflected in the fact, that the added ghost terms have a residual gauge invariance under Ω^{p-2}. And just as the Faddeev–Popov ghosts serve to get rid of the volume of the gauge group, second generation ghosts with opposite statistics (ghosts for ghosts [22]) can be used to "reintroduce" the volume of the part of the gauge group, which one had erroneously got rid of in the first step, etc.

Following Schwarz the above situation formally can be described as follows: Let the action S be a quadratic functional on the space Γ of fields φ, i.e.,

$$S(\varphi)=\langle\varphi, K\varphi\rangle, \tag{6}$$

where K is a self-adjoint operator and $\langle\ ,\ \rangle$ is some metric on Γ (in the case of a non-quadratic functional all considerations apply to the one-loop (stationary phase) approximation of the theory in question). Via this metric K can also be regarded as an operator K' from Γ to its dual Γ'.

In this setting the statement that S is degenerate (i.e., has a gauge symmetry) can be rephrased as saying that there exists an operator T_1 from some pre-Hilbert space Γ_1 to Γ, such that $K'T_1=0$. Reducibility of this gauge symmetry can then be encoded into the statement that there exists a second operator T_2 from some pre-Hilbert space Γ_2 to Γ_1 with $T_1T_2=0$, and so on.

In this way—and under the assumption that K^2 and $T_i^\dagger T_i$ are regular operators—we arrive at what Schwarz calls a resolvent of S, a sequence of pre-Hilbert spaces ($\Gamma = \Gamma_0$)

$$0 \longrightarrow \Gamma_N \xrightarrow{T_N} \Gamma_{N-1} \xrightarrow{T_{N-1}} \cdots \xrightarrow{T_1} \Gamma_0 \xrightarrow{K'} \Gamma'_0 \longrightarrow 0 \tag{7}$$

connected by linear operators satisfying $T_i T_{i+1} = 0$, which in the case $K = 0$ is just an ordinary complex. The assumption that all the operators $T_i^\dagger T_i$ are regular permits their determinant to be defined in the usual way (by zeta-function or heat-kernel regularization).

Keeping in mind the fact that in the example (1) the T_i are just exterior derivatives on the spaces Γ_i of differential forms, it is natural to define the Laplacians $\square_i$ of the complex (7) by

$$\begin{aligned} \square_0 &= K^\dagger K + T_1 T_1^\dagger \\ \square_i &= T_i^\dagger T_i + T_{i+1} T_{i+1}^\dagger. \end{aligned} \tag{8}$$

At this point it may be worth remarking that changing the metrics on the spaces Γ_i corresponds to changing the covariant (Feynman) gauge-fixing terms $T_i T_i^\dagger$, and therefore metric independence of the results is related to independence of the (covariant) choice of gauge-fixing.

By repeated application of a version of the Faddeev–Popov trick Schwarz has shown, that under the additional assumption that the sequence (7) is exact, i.e., $\mathrm{Im}\, T_{i+1} = \mathrm{Ker}\, T_i$, the partition function $Z(S)$ can be written in either of the following ways,

$$Z = \det(S)^{-1/2} \prod_{i=1}^{N} \det(T_i)^{(-1)^{i-1}} \tag{9}$$

$$Z = \prod_{i=0}^{N} \det(\square_i)^{\nu_i}, \qquad \nu_i = (-1)^{i+1} \left(\frac{2i+1}{4}\right), \tag{10}$$

where det denotes the regularized determinant and quite generally the regularized determinant of an operator T, such that $T^\dagger T$ is regular and is *defined* by $\det T = \det^{1/2}(T^\dagger T)$. Note that this definition coincides with the ordinary $\det T$ if T is itself already regular *and self-adjoint.*

In particular, in the version (10) this is recognizable as the characteristic alternating ghost for ghost [22] contribution in the partition function for anti-symmetric tensor fields and agrees with the results one would obtain from the BRST-quantization.

The assumption of exactness of (7) is, in the models to be discussed in this paper, equivalent to acyclicity of the relevant deRham complex of M. Since these models, on the other hand, are also interesting if some of the deRham cohomology groups are non-vanishing, more care has to be taken with the harmonic zero-modes of the gauge transformations (2). Before showing how this can be done (Section 2.6) we shall proceed under the assumption that no harmonic modes are present and turn to the resolvent- and BRST-computations for our BF systems next.

2.3. *Quantization of BF-Systems*

To establish the usefulness of Schwarz's formulation and its relation to the more conventional BRST-analysis let us consider first the following simple three-dimensional example:

$$S(3,1)=\int B_1\, dA_1.$$

Γ_0 is then $\Omega^1 \oplus \Omega^1$ (where $\Omega^k = \Omega^k(M)$ denotes the space of k-forms on M). The invariances of $S(3,1)$ are

$$\delta B_1 = d\Lambda_0, \qquad \delta A_1 = d\Lambda_0', \tag{11}$$

so that Γ_0 is identified as $\Omega^0 \oplus \Omega^0$ and $T_1 = d \oplus d$. There are no secondary invariances in this example and therefore the resolvent is

$$0 \longrightarrow \Omega^0 \oplus \Omega^0 \xrightarrow{T_1} \Omega^1 \oplus \Omega^1 \xrightarrow{*d} \Omega^1 \oplus \Omega^1 \longrightarrow 0. \tag{12}$$

The partition function of this theory (remember we are ignoring zero-modes) is then given by Schwarz's formula (10),

$$Z(3,1) = \det{}^{-1/4} \Box_0 \det{}^{3/4} \Box_1,$$

where $\Box_0 = \Delta_1 \oplus \Delta_1$ and $\Box_1 = \Delta_0 \oplus \Delta_0$ with Δ_k the Laplace operator on k-forms, leading to

$$Z(3,1) = \det{}^{-1/2} \Delta_1 \det{}^{3/2} \Delta_0. \tag{13}$$

The BRST-quantization of $S(3,1)$ is also quite straightforward. Since the invariances (11) are to be gauge fixed, one introduces zero-form multiplier fields E and ghosts $(\bar{c}, c)$ for the A_1 field and a similar set G and $(\bar{\omega}, \omega)$ for B_1. The gauge-fixed quantum action then is

$$S_q(3,1) = \int B_1\, dA_1 + *E\, \delta A_1 + *G\, \delta B_1 + *\bar{\omega}\, \Delta_0 \omega + *\bar{c}\, \Delta_0 c, \tag{14}$$

where δ denotes the exterior coderivative adjoint to the exterior derivative d with respect to the natural scalar product on the space $\Omega^k(M)$ determined by the Hodge duality operator $*: \Omega^k(M) \to \Omega^{n-k}(M)$. Whether δ denotes this or a variation should always be clear from the context.

Integration over the ghost fields yields $\det^2 \Delta_0$, while integration over the remaining (B, A, E, G)-system requires more care. To evaluate the determinant it is easiest to square the kinetic operator (which diagonalizes it) and then to take the square-root of the determinant of this operator. In this way one finds the contribution to be

$$\det{}^{-1/2} \Delta_1 \det{}^{-1/2} \Delta_0.$$

The result of the BRST-computation thus agrees with (13).

This method of squaring the operator which involves K and T_1 is precisely that employed by Schwarz and may be viewed as the appropriate method for disentangling the determinants that arise in the BRST-formalism.

As mentioned in the Introduction Schwarz has used actions of the form above to represent the Ray–Singer torsion. In particular it may be checked that the partition function of a BF system $S(n, n-2)$ in odd dimensions is simply the inverse of the Ray–Singer torsion, i.e., $Z(n, n-2) = T(M_n)^{-1}$, and in this example,

$$T = \det{}^{1/2} \Delta_1 \det{}^{-3/2} \Delta_0.$$

In three dimensions the abelian Chern–Simons action $\int A\, dA$ (where the A's may again take values in a flat vector bundle), on the other hand, gives a partition function Z_{CS} which is related to the Ray–Singer torsion by $Z_{CS} = T(M_3)^{-1/2}$. Thus in a sense the three-dimensional Chern–Simons system is a "square-root" of the BF system and this relation also holds for the corresponding non-Abelian actions to be discussed in Section 3. Further properties of the Ray–Singer torsion will be discussed in Section 2.5.

We now return to the important question of metric independence of the partition function, a fact which may be established by invoking Theorem 2.1 of [18], which proves just this for Ray–Singer torsion, or by following the (related) argument of Schwarz [17]. That the metric independence of the classical action is maintained at the quantum level is, however, especially easy to see in the BRST-approach, and indeed the BRST-argument can in turn be used to *prove* the metric independence of the Ray–Singer torsion!

We begin by rewriting (14) as

$$S_q = \int B_1\, dA_1 + \{Q, \Psi\}, \tag{15}$$

where

$$\Psi = *\bar{\omega}\, \delta B + *\bar{c}\, \delta A \tag{16}$$

and

$$\begin{aligned} \{Q, B\} &= d\omega \\ \{Q, \omega\} &= 0 \\ \{Q, \bar{\omega}\} &= G \\ \{Q, G\} &= 0 \end{aligned} \tag{17}$$

with identical equations for A and its ghosts and multiplier fields. Since the whole metric dependence of S_q resides in Ψ, the variation of the partition function with respect to the metric leads to

$$\frac{\delta Z}{\delta g_{\mu\nu}} = \left\langle \left\{ Q, \frac{\delta \Psi}{\delta g_{\mu\nu}} \right\} \right\rangle \tag{18}$$

which vanishes if Q is a symmetry of the vacuum. Notice that the statement of metric independence is precisely that of gauge invariance, since a change of metric may, according to (18), be compensated by a change of Ψ. The above argument straightforwardly generalizes to any Abelian BF system $S(n, p)$, since the quantum action will always be of the form $S_q(n, p) = S(n, p) + \int \{Q, \Psi\}$, and we thus have

PROPOSITION 1. *The partition function $Z(n, p)$ of any Abelian BF system $S(n, p)$ is independent of the metric.*

Alternatively we can appeal directly to [17], where Schwarz has proved rigorously the metric-independence, since in Proposition 2 below we shall show the equivalence between Schwarz's method of resolvents and the BRST-approach.

In the following we shall therefore always tacitly assume that all determinants have been regularized as in [17]. The compatibility of this regularization with BRST-invariance is analyzed in the Appendix.

2.4. *Degrees of Freedom and Corrected BRST-Counting*

From the discussion in Section 2.1 and in particular from (4) we expect to be dealing with a theory with no degrees of freedom (in the field-theoretic sense). After all the phase space is expected to be finite-dimensional if there are no "particles" present.

The situation encountered here, however, with regard to degrees of freedom of a field is quite different from that which one normally meets in gauge theories. A convenient method for counting degrees of freedom is to count the number of bosonic Laplacians in the partition function. For instance, in pure quantum electrodynamics in d dimensions, integration over the (covariantly gauge fixed) vector potential leads to $\det^{-1/2} \Delta_1$, while the ghosts contribute a factor $\det \Delta_0$. Now treating the Laplacian (solely for counting purposes) as if it acts on d copies of Ω^0 leads us to $\det^{(2-d)/2} \Delta_0$, and since $\det^{-1/2} \Delta_0$ represents one bosonic degree of freedom, the QED-answer is $d-2$.

An easy count is therefore afforded by the observation that the vector potential has d degrees of freedom, while the ghosts c and $\bar{c}$ contribute minus one degree of freedom each (this exactly corresponds to their respective contributions in terms of determinants). Formalizing this, one may count degrees of freedom for a rank n field by counting fields, ghosts, ghosts for ghosts, ... with appropriate weighting via, say, the $Osp(d/2)$ supergroup [23] (there are many alternatives).

This algorithm for counting degrees of freedom by counting fields and ghosts with the multiplicities as above is, however—as it turns out—strictly correct only for classical actions which have the conventional kinetic term quadratic in derivatives. Indeed, using this count on (14), we would be led to the conclusion that the total number of degrees of freedom for the system would be three each from A and B and minus one for each of the four ghost fields, leaving two degrees of freedom, whereas from (13) we see, in fact, that there are none.

A modified count could be that, as the operator that appears in the classical action is linear, one is overcounting by counting the fields A and B separately. Just counting the A-field (say) and all the ghosts leads us to minus one degree of freedom. The missing degrees of freedom are supplied by the multiplier fields E and G ($\frac{1}{2}$ each), as can be seen by a direct calculation. In QED the E field does not contribute to this count; here, however, it becomes dynamical. Thus while Schwarz's formulae lead us to the correct result directly, some rethinking is required in the BRST-framework.

In order to see how the BRST-counting should be modified in general it is worthwhile analyzing a reducible theory. For this purpose consider the four-dimensional model

$$S(4,2)=\int B_2\, dA_1.$$

Here $\Gamma_0=\Omega^2\oplus\Omega^1$ while the invariances are

$$\delta B_2=dA_1, \qquad \delta A_1=dA_0'$$

with a secondary invariance given by $\delta A_1=dA_0$. We have thus identified

$$\Gamma_1=\Omega^1\oplus\Omega^0, \qquad \Gamma_2=\Omega^0\oplus o, \qquad T_2=d\oplus d=T_1.$$

The resolvent is

$$0\longrightarrow\Omega^0\oplus o\xrightarrow{T_2}\Omega^1\oplus\Omega^0\xrightarrow{T_1}\Omega^2\oplus\Omega^1\xrightarrow{*d}\Omega^1\oplus\Omega^2\longrightarrow 0$$

and upon use of (10) one finds

$$Z(4,2)=\det{}^{-1/4}\Delta_2\det{}^{+1/2}\Delta_1\det{}^{-1/2}\Delta_0. \tag{19}$$

A BRST-analysis requires the introduction of ghosts for ghosts due to the secondary invariance and this corresponds to the increased length of the resolvent. One may draw a diagram corresponding to the ghost system (Fig. 1), where we have augmented the usual triangle [16] with the inclusion of the associated multiplier fields of some of the ghosts. Now conventionally the count of degrees of freedom would lead to a positive result. However, as we saw in the previous example, multiplier fields may also have to be counted. Notice that, in the BRST-action, B_2 couples to π_1 as does γ_0 (and $\bar{c}_1$ is bypassed), while A_1 couples to E_0 (and not to $\bar{\omega}_0$). The contributions of all the other fields remain the same. But now the $(B_2, \pi_1, \gamma_0, A_1, E_0)$ system is evaluated by diagonalizing as before. This gives a

a

$$\begin{array}{ccccc} & & B_2 & & \\ \pi_1 & \bar{c}_1 & & c_1 & \\ & \gamma_0 & \bar{c}_0 & & c_0 \end{array}$$

b

$$\begin{array}{ccc} & A_1 & \\ E_0 \quad \bar{\omega} & & \omega \end{array}$$

FIG. 1. (a) Ghost triangle for the B-part of the BF system $S(4,2)$ including the one multiplier field which becomes dynamical. (b) Ghost triangle for the A-part of the BF system $S(4,2)$ including the multiplier field.

$\det^{-1/4}$ of the Laplacians on each of these spaces. The A_1 triangle contribution was determined previously (it is the same here), and whereas normally B_2 would contribute $\det^{-1/2} \Delta_2$, here we see that this should be multiplied by $\det^{+1/4} \Delta_2$ (i.e., half a rank two degree of freedom should be subtracted). At the second row we need to add the contribution of π_1, namely $\det^{-1/4} \Delta_1$, while in the third row γ_0 in the old count would give $\det^{-1/2} \Delta_0$, which now gets multiplied by $\det^{+1/4} \Delta_0$. The net effect then is to subtract off half of a degree of freedom of the rank and type at each row. In the $(\bar{c}_1, c_1)$ row—though we have added half a commuting vector fields contribution—this is indeed the same as subtracting half an anticommuting vector contribution (like c_1). Taking all this into account one is led to (19).

The generalization of this result is now obvious. The length of the ghost triangle (Fig. 2), to which we have again added some of the multiplier fields, is just the length of the resolvent (7) (an observation which also facilitates the count of the unmatched zero modes in Section 2.6.).

Fields on the far left edge on every odd row (with B_p on row number one) couple to the B-system as do the multiplier fields on each even row. Therefore we see that the rule that one ought to subtract one half of the field type at each level is correct generally. In terms of partition functions this amounts to the following: Traditionally the contribution to the partition function would be

$$Z'_B = \prod_{i=0}^{p} \det{}^{\nu'_i} \Delta_{p-i}, \qquad \nu'_i = (-1)^{i+1} \frac{(1+i)}{2},$$

whereas our modified count leads us to

$$Z_B = \prod_{i=0}^{p} \det{}^{\nu_i} \Delta_{p-i}, \qquad \nu_i = (-1)^{i+1} \frac{(2i+1)}{4}.$$

$$\begin{array}{cccccccc} & & & & B_p & & & \\ & \pi_{p-1} & & \bar{c}_{p-1} & & c_{p-1} & & \\ & & \gamma_{p-2} & & \bar{c}_{p-2} & & c_{p-2} & \\ \pi_{p-3} & \gamma_{p-3} & & \phi_{p-3} & & \bar{c}_{p-3} & & c_{p-3} \\ & & & & etc.\ldots & & & \end{array}$$

FIG. 2. Ghost triangle for the B-part of the general BF system $S(n, p)$, with some of the multiplier fields influencing the count of degrees of freedom also displayed.

The action (1) of course not only includes B_p but also A_{n-p-1}, and the modified count is also to be applied to its ghost triangle, i.e.,

$$Z_A = \prod_{j=0}^{n-p-1} \det{}^{\nu_j} \Delta_{n-p-1-j}.$$

Now by Hodge duality we have $*\Delta = \Delta *$, implying $\det \Delta_k = \det \Delta_{n-k}$, so that

$$Z_A = \prod_{j=0}^{n-p-1} \det{}^{\nu_j} \Delta_{j+p+1}.$$

The partition function of the BRST-quantized version of (1) is then just the product of Z_A and Z_B, and the fact that Schwarz's geometric approach and the more algebraic BRST-quantization are equivalent for these topological theories is the content of the following proposition. It is easy to see (and may be read off from (10)) that both methods agree for more familiar (second-order) theories.

PROPOSITION 2. *The partition function of the BRST-quantized version of the classical action* $S(n, p) = \int B_p \, dA_{n-p-1}$ *coincides with the expression* $Z = \prod_{i=0}^{n} \det^{\nu_i} \Box_i$ *with* $\nu_i = (-1)^{i+1} (2i+1)/4$ *obtained from the resolvent of* $S(n, p)$.

Proof. Assuming without loss of generality that $p \geqslant n-p-1$ the general resolvent is

$$0 \to \Omega^0 \oplus o \to \cdots \to \Omega^{2p+1-n} \oplus \Omega^0 \to \cdots \to \Omega^p \oplus \Omega^{n-p-1} \to 0.$$

Now applying (10) and keeping in mind the definition (8) then completes the proof, since the resolvent gives

$$\begin{aligned} Z &= \prod_{i=0}^{n} \det{}^{\nu_i} \Box_i \\ &= \prod_{i=0}^{p} \det{}^{\nu_i} \Delta_{p-i} \prod_{j=0}^{n-p-1} \det{}^{\nu_j} \Delta_{n-p-1-j}, \end{aligned} \tag{20}$$

which coincides with the expression $Z_A Z_B$ derived above.

It is now a straightforward matter to establish that the partition function (20) correctly reproduces the result (anticipated by (4)) that the number of physical degrees of freedom of the field theory described by (1) is zero:

PROPOSITION 3. *The degrees of freedom revealed by the partition function* Z *of* $S_q(n, p)$ *add up to zero.*

Proof. The number of degrees of freedom $\sharp$ present in (20) is (recall that $\det^{-1/2} \Delta_0$ represents one bosonic degree of freedom)

$$\begin{aligned} -\sharp &= 2 \sum_{i=0}^{p} \nu_i \binom{n}{p-i} + 2 \sum_{j=0}^{n-p-1} \nu_j \binom{n}{n-p-1-j} \\ &= 2 \sum_{i=0}^{n} \nu_i \binom{n}{i} \end{aligned} \tag{21}$$

(using $\nu_{-j} = \nu_{j-1}$), which is seen to be zero upon use of the binomial identities

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

$$\sum_{i=0}^{n} (-1)^i i \binom{n}{i} = 0.$$

There are then no physical particle states and the phase space is finite-dimensional.

It should be noted that often in supersymmetric theories $Z \sim 1$ as far as the determinants are concerned. The difference here is that ghost fields are not physical and so are to be counted in a different manner to spinors.

2.5. *Path Integrals, BRST-Symmetry, and Properties of the Ray–Singer Torsion*

It is well known that the path-integral encodes a great deal of information about determinants and eigenvalues. In this section we want to show how simple scaling properties of the path-integral can be used to obtain relations between determinants of Laplace operators acting on forms of different rank. In particular, we shall as a consequence of these relations be able to prove the triviality of the Ray–Singer torsion in even dimensions (Theorem 2.3 of [18]) directly from properties of the path-integral.

For example, in two dimensions it may easily be proven (we still assume absence of harmonic modes, i.e., acyclicity of the deRham complex of the forms under consideration) that $\det f(\Delta_1) = \det^2 f(\Delta_0)$, where f is some function of the Laplace operator (e.g., $f(\Delta) = \Delta$). This is seen by considering the following partition function:

$$Z = \int \mathscr{D}A \, \mathscr{D}B \, \mathscr{D}E e^{\int B dA + E d * A}.$$

The action is invariant under the following scaling of the fields: $A \to f(\Delta_1) A$, $B \to f^{-1}(\Delta_0) B$, $E \to f^{-1}(\Delta_0) E$, and since Z cannot be changed by this transformation, the Jacobian of this transformation must be equal to one. This yields the desired result.

In odd dimensions this procedure does not give us any information. Let us, for instance, consider the three-dimensional example (14) of Section 2.3. The transformations that leave the action invariant are

$$A \to f(\Delta_1)A, \qquad B \to f^{-1}(\Delta_1)B$$
$$E \to f^{-1}(\Delta_0)E, \qquad G \to f(\Delta_0)G$$

whose total Jacobian is identically one. It is, however, precisely this extra information that we have at our disposal in even dimensions which allows us to prove the triviality of the Ray–Singer torsion in that case. In two dimensions

$$T(M_2) = \det{}^{-1/2} \Delta_1 \det \Delta_2 = \det{}^{-1/2} \Delta_1 \det \Delta_0 = 1$$

as a consequence of duality and the above result. Generalizing these considerations one finds

PROPOSITION 4. *Let the deRham cohomology ring $H^*(M)$ of forms on an even $(n = 2m)$-dimensional orientable compact manifold M without boundary be trivial. Then the determinant of any function of the Laplace operator on m-forms can be expressed in terms of the determinants of the Laplace operator acting on lower rank forms*:

$$\det f(\Delta_{n/2}) = \prod_{i=0}^{n/2-1} \det{}^{2(-1)^i} f(\Delta_{n/2-i-1}). \tag{22}$$

Proof. Let $S_q(n, p)$ be the full BRST-extended quantum action corresponding to the classical action (1). If one scales B_p by a factor $f(\Delta_p)$ in the path-integral

$$\int \mathscr{D}[\text{allfields}]\, e^{S_q(n,p)},$$

this may be compensated in the first term of the action (which is just the classical action) by scaling A_{n-p-1} by $f^{-1}(\Delta_{n-p-1})$. All other fields that appear can then also be scaled in such a way that one returns to the original action. The product of determinants obtained in this way must therefore equal one. This implies

$$\begin{aligned} 1 &= \det f(\Delta_p) \det{}^{-1} f(\Delta_{p-1}) \cdots \det{}^{(-1)^p} f(\Delta_0) \\ &\quad \times \det{}^{-1} f(\Delta_{n-p-1}) \det f(\Delta_{n-p-2}) \cdots \det{}^{(-1)^{n-p}} f(\Delta_0), \end{aligned} \tag{23}$$

where we have collected the contributions from the fields coming from the B- and A-triangles in the first and second rows respectively. For n odd (23) is identically satisfied because of Hodge duality, whereas for n even the two sets of terms do not cancel but rather add up upon using duality. Collecting all the terms together one then arrives at (22).

As a consequence of this proposition we can now prove

COROLLARY 1 (Ray and Singer). *Suppose M is an oriented compact manifold without boundary, of even dimension. Then* $\log T(M) = 0$.

Proof. In even dimensions, $n = 2m$,

$$T(M) = \prod_{q=0}^{2m} \det{}^{(-1)^q q/2} \Delta_q. \tag{24}$$

Extracting the term proportional to $\det \Delta_m$ from (24) and using duality on the remainder to express everything in terms of $\det \Delta_k$'s with $k \leqslant m-1$, one arrives at

$$T(M) = \det{}^{(-1)^m m/2} \Delta_m \prod_{q=0}^{m-1} \det{}^{m(-1)^q} \Delta_q, \tag{25}$$

which is equal to one by Proposition 4.

2.6. *Treatment of Harmonic Zero-Modes*

So far we have—as repeatedly emphasized above—assumed that there are no zero-eigenvalues of the Laplace operator Δ_k on the space $\Omega^k(M)$ of k-forms on M with values in a flat vector bundle. By Hodge's theorem this means that we have been assuming that all the deRham cohomology groups $H^k(M) \equiv H^k$ are trivial.

Looking back at (1) and the discussion following it, we see that in this case all solutions to the equations of motion (3) are gauge-equivalent to zero, which implies that the reduced phase space $\mathcal{N}$ of (4) is just a point. We shall now show how to incorporate these so-far neglected zero-modes into our discussion and thereby promote the point to a finite-dimensional phase space of the form $H^p \oplus H^{n-p-1}$, providing us with a field theoretic model for the deRham complex of a manifold.

We shall show below how this is accomplished within the path-integral using the Faddeev–Popov procedure also to gauge the harmonic parts appearing in the Hodge decomposition of the fields. We shall then show that this is equivalent to the more commonly used prescription of "inserting zero modes into the path-integral measure." Before doing so, however, let us try to understand geometrically what is required here within the framework of Schwarz's approach explained above [24].

Schwarz's definition of the partition function (10) may be regarded as the instruction to choose as quantum action appearing in the path-integral definition of the partition function the action

$$S_q = S(f_0) + \langle T_1 f_1, T_1 f_1 \rangle + \cdots, \tag{26}$$

where the variables f_i are Grassmann even (odd) for i even (odd), and the domain of integration for f_i is the orthogonal complement to Im T_{i+1} in Γ_i. Decomposing Γ_i as

$$\begin{aligned}\Gamma_i &= \mathrm{Im}\ T_{i+1} \oplus \Gamma_i^{\perp} \\ &= \mathrm{Im}\ T_{i+1} \oplus \mathrm{Ker}\ T_{i+1}^{\dagger} \\ &= \mathrm{Im}\ T_{i+1} \oplus \mathrm{Im}\ T_i^{\dagger} \oplus h_i,\end{aligned} \tag{27}$$

where

$$h_i = \mathrm{Ker}\ T_{i+1}^{\dagger}/\mathrm{Im}\ T_i^{\dagger} = \mathrm{Ker}\ T_i/\mathrm{Im}\ T_{i+1},$$

one finds the domain of integration of f_i to be

$$\Gamma_i^{\perp} = \mathrm{Im}\ T_i^{\dagger} \oplus h_i \tag{28}$$

and (10) is, in fact, the result obtained upon integration of (26) over all the Im $T_i^{\dagger} \subseteq \Gamma^{\perp}$. Thus the (finite-dimensional) integral over the cohomology groups h_i remains to be done.

In the context of our BF systems the above amounts to saying that one has performed the path-integral over the coexact pieces of all the fields involved, the exact pieces having been taken care of by the usual gauge fixing prescription, while the integral over the cohomology groups H^i (with alternating Grassmann parity) still remains. Before showing how this can be done in the BRST-framework let us clarify a seeming discrepancy between the BRST-approach and the approach of Schwarz.

In the ghost triangle many fields appear which do not fit into the resolvent of Schwarz (more precisely the fields in the resolvent correspond to the fields on the right edge of the ghost triangle). We have seen above that this does not prevent the partition functions from agreeing. But what about the zero-modes of fields appearing in the ghost triangle which have no counterpart in the resolvent? Do they or do they not play a role? Fortunately they do not, since they all have superpartners (i.e., $\{Q, \bar{c}_k\} = \pi_k$). Thus their zero-modes match precisely and—having opposite statistics—do not influence the path integral.

Thus to count unmatched modes one simply starts at the top of the triangle and counts cohomology groups along the right edge with alternating sign (symbolically, that is) corresponding to the alternating Grassmann parity of their respective representatives. This means that we get $H^p \ominus H^{p-1} \oplus H^{p-2} \ominus \cdots$ (where p is the rank of the classical field). It is important to realize that this count does not depend on the starting action in the sense that, e.g., $B_2\, dA_1$ is as good as $\frac{1}{4} dA_1 * dA_1$, but the count must be made on the triangles of all the classical fields present.

Thus, for example, in the three-dimensional BF system, $S(3, 2) = \int B_2\, dC_0$, B_2 leads to a zero mode contribution to the path integral of the form $H^2 \ominus H^1 \oplus H^0$, while C_0 contributes H^0. We are, therefore, led to the conclusion that the remaining zero mode integration needs to be performed over $H^2 \ominus H^1 \oplus 2H^0$. Likewise, the Abelian Chern–Simons action $\int A_1\, dA_1$ leads to $H^1 \ominus H^0$, whereas its "square," the BF system $S(3, 1)$ which was discussed in Section 2.3, gives rise to $2H^1 \ominus 2H^0$. Generalizing this we find

PROPOSITION 5. *The path-integral treatment of BF systems leads to the following* graded *sum of cohomology groups for the zero-mode contribution: For the BF system,* $S(n, p) = \int B_p\, dA_{n-p-1}$ *in n dimensions,*

$$\sum_{i=0}^{2p-n} (-1)^i H^{p-i} \oplus (1 + (-1)^{n+1}) \sum_{i=2p-n+1}^{p} (-1)^i H^{p-i} \tag{29}$$

and for the BF system $\int A_p\, dA_p$ *(i.e., the Abelian Kalb–Ramond Chern–Simons action) in* $2p+1 = 4k+3$ *dimensions (in* $4k+1$ *dimensions the action is zero),*

$$\sum_{i=0}^{p} (-1)^i H^{p-i}. \tag{30}$$

Furthermore, this is in agreement with the conclusions drawn from the resolvent approach.

We shall now give a prescription to perform this remaining integral which is particularly transparent in the BRST-framework. Usually, to be able to do calculations with the path-integral in the presence of fermionic (i.e., Grassmann odd) zero-modes, one needs to insert the appropriate number of fields to ensure non-vanishing of expectation values (as a consequence of the rules of Berezinian integration). For bosonic zero-modes, on the other hand, there is an infinity problem (after all, bosonic determinants appear in the denominator and thus lead to infinities if there is a zero eigenvalue).

In that case there are two possibilities available. The first is to accept them and to use the path integral measure over these variables as an appropriate one for calculating expectation values of certain functions on the zero-mode space.

Alternatively (for both even and odd zero modes), thinking of the zero modes as representing gauge degrees of freedom, it is also possible to gauge-fix the theory in such a way from the outset that the resulting partition function is well defined. For fermions this automatically "inserts" the appropriate number of fermions into the measure to soak up the zero-modes, rendering the path-integral finite (not zero).

This approach of treating zero-modes—or collective coordinates—as a gauge-fixing problem was advocated by Polyakov [25] and formalized in the BRST language by Amati and Rouet [26] and—using Ward identities—by Babelon [27]. It relies on straightforward use of the Faddeev–Popov procedure, so consequently the gauge-fixed action has a very simple BRST-invariance.

The analogy with ordinary gauge invariance is that in QED (say) the part of the vector potential A which lies in the gauge direction does not enter into the action and is the cause of the problems associated with defining the partition function. This is precisely the situation we are confronted with in the presence of zero-modes. Using the Hodge decomposition (or (27) in the general case) of a p-form,

$$A_p = \delta\alpha_{p+1} + d\beta_{p-1} + \gamma_p, \tag{31}$$

where γ_p is harmonic, we can read off what the appropriate gauge-fixing should be.

When the action is invariant under $A_p \to A_p + d\Lambda_{p-1}$ this means that β_{p-1} does not appear. Gauge-fixing then amounts to projecting β_{p-1} out by means of a Lagrange multiplier enforcing a delta function constraint on A in the path integral. Thus one adds a term to the action which does precisely this—$\langle dE, A \rangle$—plus the corresponding ghost terms.

Now the zero mode problem is posed as the invariance of the action under $A_p \to A_p + \Lambda_p$, where Λ_p is harmonic, which means that γ_p of (31) does not enter into the action. Following the previous rationale we gauge-fix by projecting out γ_p, i.e., by adding a term $\langle \Sigma, A \rangle$ to the action, where Σ is an arbitrary harmonic form. The Faddeev–Popov determinant one gets in this way is $\det v$, where v is the volume of the compact manifold M. The gauge-fixed part of the action may then be written as

$$\varepsilon^i \int \lambda_i \cdot A + v\bar{c}^i c^i, \qquad i = 1, ..., b_i = \dim H^i(M), \tag{32}$$

where $\Sigma = \varepsilon^i \lambda_i$, λ_i chosen to be an orthogonal basis of harmonic p-forms,

$$\int \lambda_i \lambda_j = v\delta_{ij}.$$

The action then possesses the following BRST-symmetry (the "fields" c, $\bar{c}$, and ε are not functions of space-time!):

$$\begin{aligned} \delta A &= \lambda^j c_j \\ \delta c_j &= 0 \\ \delta \bar{c}_j &= \varepsilon_j \\ \delta \varepsilon_j &= 0 \\ \delta^2 &= 0. \end{aligned}$$

Unlike the conventional gauge invariance for p-forms we do not find that this ghost system entails ghosts for ghosts or ghost interactions (harmonic forms are their own Hodge decomposition).

As an example consider the two-dimensional Lagrangian for the chiral (i.e., we only consider the term involving ∂_-) b-c system on a Riemann surface M_g of genus $g \geqslant 2$

$$S = \int c^+ \partial_- b_{++}.$$

There are then no c^+ zero modes and $3g-3$ b_{++} zero-modes. The corresponding gauge fixed action is

$$S_q = S + \varepsilon^i \int \lambda_i^{++} b_{++} + v\bar{c}^i c_i,$$

where $i = 1, \ldots, 3g - 3$. The ε^i are anticommuting, while $(\bar{c}^i, c_i)$ are Grassmann even. The partition function is

$$Z = \int \mathscr{D}c^+ \, \mathscr{D}b_{++} \, \mathscr{D}\varepsilon^i \, \mathscr{D}\bar{c}^j \, \mathscr{D}c^k e^{S_q}$$

$$= \int \mathscr{D}c^+ \, \mathscr{D}b_{++} e^S \prod_{i=1}^{3g-3} \int \lambda_i^{++} b_{++}$$

(where the second line follows from the first by expanding the exponential). Thus we see explicitly that gauge fixing is equivalent to the insertion of the appropriate number of modes.

The method outlined above works well for the fields that are present on the right-hand ledge of the ghost-triangle. For these fields the complete BRST-transformation is of the form

$$\{Q, c_p\} = \lambda_p^j c_j + dc_{p-1}$$

$$\{Q, c_j\} = 0,$$

giving as usual $Q^2 = 0$. As remarked earlier, for ghosts that do not lie on the right-hand ledge of the ghost diagram, an alternative procedure presents itself; let us make that statement precise here. These fields come in pairs $(\bar{c}_k, \pi^k)$ with the BRST-transformation rules

$$\{Q, \bar{c}_k\} = \pi_k, \qquad \{Q, \pi_k\} = 0.$$

There is no point in adding the zero mode shifts explicitly in these formulae, since this would just amount to a redefinition of the π_k. Instead, add the ghosts $\bar{\sigma}^i$, τ^i satisfying

$$\{Q, \bar{\sigma}^i\} = \tau^i, \qquad \{Q, \tau^i\} = 0$$

and add to the action the BRST exact term

$$\{Q, \bar{\sigma}^i \lambda_i^k \bar{c}_k\}$$

which represents the gauge-fixing of both sets of zero-modes simultaneously. In terms of the signed sums of cohomology groups this means that the $\bar{c}_k$ contribute $\pm H^k$, while the π_k yield $\mp H^k$, the sum being zero.

So far we have dealt with the situation where one wishes to gauge-fix the zero-mode contributions to zero. In some instances it may be more desirable to keep the zero-mode integration explicit, as—say—in Witten's topological Yang–Mills theory [1], or as in the two-dimensional non-Abelian theory to be discussed in Section 3.2 below. The question that naturally arises is, how to decompose the fields into their harmonic and non-harmonic pieces in a BRST well-defined way. In this case decompose a p-form A_p as

$$A_p = A_p^q + A_p^c(\lambda),$$

where A_p^q is the "quantum" part and A_p^c is the corresponding classical part. The aim is to gauge-fix A_p^q in such a way that all components in the moduli space are projected out. Let

$$\delta\lambda^i = \sigma^i, \qquad \delta A_p^q = -\sigma^i \partial_i A_p^c(\lambda)$$

so that $\delta A_p = 0$. The complete BRST operator is thus taken to be $Q + \delta$. To project out the components of A_p^q that lie in the harmonic directions one makes use of the above symmetry. Introducing extra fields $(\bar{\sigma}^i, \tau^i)$ with Q-transformations as above, an appropriate addition to the action is

$$\int \{Q + \delta, \bar{\sigma}^i A_p^q A_{p,\,i}^c(\lambda)\}.$$

Once more for the natural pairs of fields that lie off the right-hand ledge of the ghost triangle, the situation can be simplified. The idea is to make use of the classical/quantum split as above, but to relate the classical components of the two members of the pair under the BRST transformation. Thus one splits $\bar{c}_k$ and π_k into their quantum and zero-mode parts,

$$\bar{c}_k = \bar{c}_k^q + \psi_i \gamma_k^i$$
$$\pi_k = \pi_k^q + \phi_i \gamma_k^i,$$

where γ_k^i form a basis of the cohomology group H^k and where one may take

$$\{Q, \psi_i\} = \phi_i,$$

in this way dealing with both fields at once. Adding anti-ghosts $\bar{\psi}_i$ and multipliers χ_i in the usual manner to complete the BRST algebra, an appropriate addition to the action is

$$\{Q, \bar{\psi}_i \bar{c}_k^q \gamma^{ik}\}.$$

A general benefit of these procedures is that once more there is no spurious metric dependence introduced by the gauge fixing, since again the change in Z induced by a change in the metric is the vacuum expectation value of a BRST-variation and hence zero.

3. Non-Abelian Models

3.1. *General Aspects*

In this section we shall analyze in some detail an obvious but—as it will turn out—interesting and rich generalization of the BF systems we have discussed so far, which amounts to replacing the Abelian gauge group of the models of the previous section by a non-Abelian one. This is accomplished by replacing dA_1 by $F=F_A$, the curvature two-form of a connection A on a principal bundle P over M with (simple) structure group G. B will then have to be a section of $\Omega^{n-2}(M, \text{ad}\, P)$, i.e., locally a Lie algebra valued form on M transforming under the adjoint representation of G. Our action will then be

$$S=\int \operatorname{tr}(BF_A) \tag{33}$$

and of particular interest to us will be the two-dimensional versions of these non-Abelian BF systems which are—in a precise sense, as we shall show below—the natural analogue of the Chern–Simons action in three dimensions. They allow us, for instance, to explicitly write down a Nicolai map [8] which trivializes (up to zero-modes, of course) not only these BF systems, but also true two-dimensional Yang–Mills theory in arbitrary covariant gauges (Section 3.2).

Moreover, these systems will provide us with a two-dimensional theory of topological gravity (Section 3.3) giving rise to a field-theoretic description of the moduli spaces of Riemann surfaces. This description bears some striking resemblances to Hitchin's [19] approach to Teichmüller space via the dimensionally reduced self-duality equations, and we will attempt to sharpen this analogy by recalling the pertinent features of his work.

In addition to the usual gauge invariance,

$$\begin{aligned} \delta A &= d_A \Lambda \\ \delta B &= [B, \Lambda], \end{aligned} \tag{34}$$

one has (as a consequence of the Bianchi identity) the non-Abelian $p=(n-2)$-form symmetry

$$\begin{aligned} \delta A &= 0 \\ \delta B &= d_A \Phi \end{aligned} \tag{35}$$

in dimensions greater than two. The action (33) leads to the equations of motion

$$\begin{aligned} F_A &= 0 \\ d_A B &= 0 \end{aligned} \tag{36}$$

which imply that the symmetry (35) is on-shell reducible in four or more dimensions. This leads to some complications in the process of quantization which is therefore most conveniently accomplished by means of the Batalin–Vilkovisky algorithm [16]. For a detailed discussion of the four-dimensional case we refer to Section 3.4.

The non-Abelian BF systems in two and three dimensions may be quantized directly by standard BRST-techniques. This implies in particular that our argument of Section 2 concerning the metric independence of the corresponding partition function carries over directly to these theories. For higher-dimensional systems it is more difficult to establish their topological nature, and a complete proof may have to await a fuller elucidation of the Batalin–Vilkovisky procedure itself. Some arguments along this line and a proof of metric independence in four dimensions may be found in Section 3.4 as well as various pieces of further circumstantial evidence.

Another class of potentially interesting actions for topological field theories with non-Abelian symmetries is provided by the Chern–Simons functionals. As discussed in the Introduction only the three-dimensional version

$$S_{CS}(A) = \int_{M_3} \operatorname{tr}\left(A\, dA + \frac{2}{3} A^3\right) \tag{37}$$

has so far been seriously considered as (part of) an action functional, due to the higher-derivative character of the higher-dimensional counterparts even in the Abelian case. This action has been studied in some detail recently by Witten [3, 15], and the remarkable properties of (37) he has uncovered may make the following observation concerning the intimate relationship between the Chern–Simons action (37) and the action of the non-Abelian BF system (33) in two dimensions of some interest.

If the three-manifold M_3 appearing in (37) is of the form $M_3 = M_2 \times S^1$, where M_2 is some orientable boundaryless two-manifold and S^1 is the circle, one can dimensionally reduce the Chern–Simons action to an action on M_2 by assuming A to be independent of the coordinate on the circle and performing the integral over S^1. In this way one arrives at a metric-independent action in two dimensions which turns out to be the two-dimensional non-Abelian BF system (upon identification of B with the coefficient of A along the circle). Higher-dimensional analogues of this relation exist only in the Abelian case. In $4k+3$ dimensions one can consider the action

$$S = \int A_{2k+1}\, dA_{2k+1} \tag{38}$$

which upon dimensional reduction as above yields the Abelian BF system $S(4k+2, 2k)$. The would-be candidate in $4k+1$ dimensions, however, vanishes, due to the fact that the fields appearing in the action are forms of even rank, unless one is willing to regard them as anticommuting objects by equipping them with an additional Grassmann odd grading. The most trivial situation where this possibility occurs is in one dimension, where it leads to the usual fermionic action

$$S(\psi) = \int \psi\dot{\psi}. \tag{39}$$

Whether a generalization of this to topological field theories with a priori anticommuting variables is interesting remains to be seen. However, it should be noted that these theories may also be formulated in any dimension, and—when used to calculate intersection numbers in Section 4—lead to essentially the same results as the bosonic theories.

3.2. *BF Systems, Nicolai Maps, and Yang–Mills Theory in Two Dimensions*

The action of the BF system in two dimensions (33) becomes upon BRST-gauge fixing of the symmetry (34) ((35) being absent in two dimensions)

$$S_q = \int \operatorname{tr}(BF_A + *E\,\delta A + d\bar{\omega} * d_A\omega), \tag{40}$$

where E and $(\bar{\omega}, \omega)$ are the scalar multiplier field and the ghosts, respectively. Metric independence of the partition function is guaranteed, since—as in the abelian theories—S_q may be written as

$$S_q = S + \int \{Q, \bar{\omega}\,\delta A\}, \tag{41}$$

where Q is the conventional non-Abelian BRST-operator. The prescription for counting degrees of freedom explained in Section 2.4 is valid here (since these are unaffected by the presence of the interaction-term in (33)) and leads to the conclusion that the configuration space is finite-dimensional.

This can be made more explicit by using the equations of motion (36) and the gauge symmetries (34). From the B-equation of motion we learn that the connection A has to be flat, whereas the A-equation of motion then tells us (taking into account the symmetries (34)) that the reduced phase space $\mathcal{N}$ is a fibre bundle over the moduli space $\mathcal{M}$ of flat connections, with fibre over a point A of $\mathcal{M}$ being the space $\mathcal{B}_A$ of gauge equivalence classes of covariantly constant B's, i.e., locally

$$\mathcal{N} = \mathcal{M} \times \mathcal{B} \tag{42}$$

which is the non-linear generalization of the reduced phase space (4) of the Abelian BF system. Indeed the tangent space to $\mathcal{N}$ at a point (of equivalence classes of) (A, B) is the precise analogue of (4),

$$T_{(A,B)}\mathcal{N} = H^0_A(M, \operatorname{ad} P) \oplus H^1_A(M, \operatorname{ad} P), \tag{43}$$

where

$$H^i_A(M, \operatorname{ad} P) = \frac{\operatorname{Ker} d_A : \Omega^i(M, \operatorname{ad} P) \to \Omega^{i+1}(M, \operatorname{ad} P)}{\operatorname{Im} d_A : \Omega^{i-1}(M, \operatorname{ad} P) \to \Omega^i(M, \operatorname{ad} P)}, \tag{44}$$

which makes sense since $(d_A)^2 = 0$ for A, a solution to the equations of motion.

We are thus led to suspect that the expression for the partition function of (40) reduces to a finite-dimensional integral over $\mathcal{M}$, and in this two-dimensional example this can be done quite explicitly by means of a Nicolai map [8, 9]: The change of variables in (40), $A \to (\xi, \eta)$, defined by

$$\begin{aligned} \xi(A) &= F_A \\ \eta(A) &= \delta A, \end{aligned} \tag{45}$$

produces the Jacobian

$$J = \left| \det \frac{\partial A}{\partial(\xi, \eta)} \right| = |\det \delta d_A|^{-1} \tag{46}$$

which cancels precisely the determinant arising upon integration over the ghosts in (40), leaving just a trivial Gaussian integral over the variables (B, E, ξ, η).

The existence of this Nicolai map, however, by no means implies that the theory is trivial. Indeed we see from (45) that the transformation to the variables (ξ, η) permits us to trivialize the path-integral for the partition function over all but a finite-dimensional subspace $\mathscr{M}'$ of fields, points of which are—modulo questions concerning the possibility of fixing the gauge globally in field space—in one-to-one correspondence with points in the moduli space $\mathscr{M}$! Thus—as expected—we are only left with a path-integral over the finite-dimensional reduced configuration space $\mathscr{M}$.

To see the how to properly handle the zero-modes in this example we generalize the analysis of Section 2.6. Write

$$A = A_q + A_c(\lambda),$$

where $F_{A_c} = 0$ and $d_{A_0} * (A_c(\lambda) - A_0) = 0$ $(A_0 = A_c(0))$, so that λ parameterizes $H^1(M, \mathrm{ad}\, P)$. Also for simplicity assume $H^0(M, \mathrm{ad}\, P) = 0$, so that the connections are irreducible. The techniques developed here can also be applied in the more general setting when reducible connections are present. Now take

$$\begin{aligned} &\{Q, \lambda^i\} = \sigma^i, && \{Q, \sigma^i\} = 0 \\ &\{Q, \bar{\sigma}^i\} = \tau^i, && \{Q, \tau^i\} = 0 \\ &\{Q, A\} = d_A \omega && \end{aligned}$$

and choose the quantum action to be

$$\begin{aligned} S &= \int BF_A + \{Q, \bar{\omega} d_{A_0} * (A - A_0) + \bar{\sigma}^i A_q \, \partial_i A_c(\lambda)\} \\ &= \int BF_A + *E \, d_{A_0} A_q + d_{A_0} \bar{\omega} * d_A \omega + \tau^i A_q \, \partial_i A_c(\lambda) \\ &\quad + \bar{\sigma}^i \sigma^j \, \partial_j A_c(\lambda) \, \partial_i A_c(\lambda) - \bar{\sigma}^i \, d_A \omega \, \partial_i A_c(\lambda) - \bar{\sigma}^i A_q \sigma^j \, \partial_i \, \partial_j A_c(\lambda). \end{aligned}$$

The integrations over B, E, and τ lead to delta-functions that set A_q to zero. This in turn implies that the last two terms in the action vanish. At this point it is now a straightforward exercise to show that the Jacobian that arises (defined and regularized as in [17]) exactly cancels the ghost determinant (also regularized via Schwarz's method).

The final result is thus an integration over the moduli space, in agreement with the argument based on the Nicolai map. While the Nicolai map seems to be less rigorous than the explicit introduction of the zero modes, it provides us with an alternative and intuitive picture of how the partition function reduces to an integral over the moduli space.

To proceed with the evaluation of the partition function we have to obtain some more information on $\mathcal{M}$. As a first step a simple estimate on its dimension may be obtained as follows: Via its holonomy a flat connection A on P gives rise to a homomorphism from the fundamental group $\pi_1(M)$ of M to G, uniquely defined up to conjugation by elements of G. Conversely, given such a homomorphism h, we may construct a principal G-bundle with a flat connection whose holonomy is described by h, by associating it to the principal $\pi_1(M)$-bundle $\tilde{M} \to M$ (where $\tilde{M}$ is the universal covering manifold of M) via h. Thus $\mathcal{M}$ may be identified with the space of homomorphisms from the fundamental group of M to G modulo the action of conjugation by elements of G,

$$\mathcal{M} = \mathrm{Hom}(\pi_1(M), G)/G. \tag{47}$$

If, for instance, $M = M_g$ is a Riemann surface of genus $g > 1$, π_1 has $2g$ generators satisfying one relation. Thus if G is simple and compact $\mathcal{M}$ is a space of dimension

$$\dim \mathcal{M} = (2g-1)\dim G - \dim G = (2g-2)\dim G, \tag{48}$$

whereas if $G = U(1)$, the one relation is automatically satisfied and the action of $U(1)$ by conjugation is trivial, and hence

$$\dim \mathcal{M} = 2g \tag{49}$$

in this case. This simply corresponds to assigning boundary conditions along the $2g$ homology cycles to charged fields or—equivalently—to a $2g$-parameter family of θ-vacua.

Returning now to our problem of defining the integral over $\mathcal{M}$ we see that we are faced with two alternatives. $\mathcal{M}$ may be zero-dimensional and consist of isolated points. This happens, for example, in the Abelian case, when the first homology group of M has torsion (as for RP^2), which implies that there are topologically non-trivial flat complex line bundles on M. In that case the partition function is a (normalized) sum over one's (one for each point), since the determinants coming from the ghost part and the Jacobian were identical. It may be, though, that a more careful treatment will reveal additional sign factors for these 1's, as happens for the isolated instanton contributions in Witten's work [1] on topological field theory.

On the other hand, $\mathcal{M}$ may be of finite non-zero dimension. Then the partition function, as it stands, is not really defined. However, treating the coordinates of $\mathcal{M}$ as collective coordinates, the Faddeev–Popov procedure can be adapted to handle this situation as well. We shall leave the details of this for future investigations. Suffice it to say here that the net effect of the extra ghost contribution is to modify the measure on $\mathcal{M}$. This feature is, of course, familiar from instanton calculations [25].

We shall now briefly leave the issue of topological field theories and turn our attention towards (the closely related as we shall see) two-dimensional Yang–Mills theory.

It has been known for a long time that great simplifications occur in the axial gauge ($A_0 = 0$) in two dimensions. 't Hooft [28] used this to great effect in his study of the large N behaviour of QCD_2 (incorporating quarks). In this gauge there are no self-interactions among the gauge fields and the ghost terms decouple. On a space-time which is not of the form $\Sigma \times R$ such a choice of gauge may not be possible. Covariant gauges, on the other hand, are allowed on any manifold (ignoring Gribov ambiguities), but the apparent drawback here is that gauge self-interactions are present and the ghosts do not decouple from the gauge fields. This makes it much harder to establish the triviality of Yang–Mills theory on two-dimensional Minkowski space in these gauges.

However, using the Nicolai map introduced for the BF system above we shall now show that the partition function of two-dimensional Yang–Mills theory is exactly calculable in arbitrary gauges (and indeed for arbitrary surfaces). To that end, consider the action

$$S = \int \operatorname{tr}\left(BF_A - \frac{1}{2} B * B\right) \tag{50}$$

which is clearly no longer metric independent. The equations of motion following from S are

$$\begin{aligned} F_A &= *B \\ d_A B &= 0. \end{aligned} \tag{51}$$

Combining these, one finds precisely the Yang–Mills equation

$$d_A * F_A = 0; \tag{52}$$

while substituting (51) into (50) (or, alternatively, performing the Gaussian integral over B) one sees that S becomes just the usual Yang–Mills action

$$S = \frac{1}{2} \int \operatorname{tr}(F_A * F_A). \tag{53}$$

Notice that, since (50) differs from the action of the two-dimensional BF system just by a term proportional to B^2, the Nicolai map (45) still reduces the bosonic part of the path-integral to a Gaussian, while its Jacobian still cancels against the contribution from the ghosts, since the gauge-fixing and ghost kinetic terms for (50) and the BF system are identical.

Thus the partition function of two-dimensional Yang–Mills theory is now expressed as the integral over the same finite-dimensional moduli space $\mathcal{M}$ as before. This does, of course not by itself, imply that the phase spaces of the theories are identical. And, indeed, expectation values of observables (like F_A, for instance) may well be (and are!) different. Furthermore, they will, in general, receive contributions from different parts of field space, since the Nicolai map will, in general, not be available to trivialize correlation functions.

If M is a cylinder, however (and thus the only non-trivial two-dimensional surface admitting a space-time interpretation), the fact that $\mathcal{M}$ can nevertheless be identified with the phase space of two-dimensional Yang–Mills theory follows from recent work of Rajeev [20], who elegantly determined explicitly the reduced phase space of this system. Rajeev's result is that if the gauge group G is simply connected and compact, the phase space $\mathcal{N}'$ of two-dimensional Yang–Mills theory on a cylinder is

$$\mathcal{N}' = (G \oplus L(G))/\text{ad}\, G, \tag{54}$$

where $L(G)$ is the Lie algebra of G and the quotient is taken with respect to the adjoint action of G on $L(G)$ and itself. On the other hand, our moduli space of flat connections is

$$\mathcal{M} = \text{Hom}(\pi_1(M), G)/\text{ad}\, G = \text{Hom}(Z, G)/\text{ad}\, G = G/\text{ad}\, G; \tag{55}$$

i.e., $\mathcal{M}$ is just the space of conjugacy classes of G. Thus rewriting $\mathcal{N}'$ as

$$\mathcal{N}' \approx (G \oplus L(G)^*)/\text{ad}\, G \approx T^*G/\text{ad}\, G \approx T^*(G/\text{ad}\, G), \tag{56}$$

we indeed find

$$\mathcal{N}' \approx T^*\mathcal{M} = \mathcal{N}, \tag{57}$$

as claimed.

3.3. *Topological Gravity and the Self-Duality Equation on a Riemann Surface*

Combining the strategy of earlier attempts [29] with an adaptation of the gauge theoretic formulation of three-dimensional Einstein gravity [15] Montano and Sonnenschein [30] have recently suggested a theory of topological gravity in two dimensions which leads in a fairly straightforward manner—paralleling the interpretation of the Donaldson polynomials [13] as observables of a topological quantum field theory [1]—to the construction of observables on the moduli spaces of Riemann surfaces. Their theory is (for genus greater than one) a topological $SO(2, 1)$ gauge theory in the sense of [1], where the zweibein and the spin-connection appear as a priori independent coefficients of the $SO(2, 1)$-connection. The huge topological symmetry of the theory, however, permits them to impose the vanishing of its curvature as a gauge condition, which enforces the spin-connection to be torsion-free and furthermore requires the two-dimensional metric to be of constant negative curvature (-1). Since for all Riemann surfaces of genus greater than one there is precisely one such metric in each conformal class, this establishes the relation to the moduli space of Riemann surfaces.

In this section we wish to present an alternative approach to two-dimensional topological gravity which is based on a non-Abelian BF system with the same gauge group $SO(2, 1)$. This construction—though on the question of invariants on the moduli space is not as clear as [30] (see Section 4)—seems to offer some advantages over that proposal, the most obvious perhaps being that the vanishing curvature condition (with the same implications as above) arises naturally as an equation of motion instead of as a gauge condition. Its quantization is very simple and another nice feature is that we can also interpret this theory as a classical theory of gravity, whereas the model of Montano and Sonnenschein is purely quantum in nature.

Furthermore—and perhaps most importantly—this theory of topological gravity displays some striking analogies with Hitchin's approach to moduli space [19] which we shall explain below after having discussed the two-dimensional gravity BF system itself. For simplicity we restrict our attention to Riemann surfaces of genus greater than one.

Our basic object of interest will be the $SO(2, 1)$-connection A which we write as

$$A = \omega J + e^a P_a, \qquad a = 1, 2, \tag{58}$$

where the coefficients are ultimately to be identified with the spin-connection and zweibeins, respectively. The generators (J, P_a) of the Lie algebra satisfy the commutation relations

$$\begin{aligned} [P_a, P_b] &= \varepsilon_{ab} J \\ [J, J] &= 0 \\ [J, P_a] &= \varepsilon_a{}^b P_b. \end{aligned} \tag{59}$$

The Killing–Cartan metric is in this basis,

$$\begin{aligned} \langle J, J \rangle &= 1 \\ \langle P_a, P_b \rangle &= \delta_{ab}, \end{aligned} \tag{60}$$

and in addition to A we have to introduce another (scalar) field B which will, however—as long as we are not discussing observables (Section 4)—play a somewhat passive role. With all this in mind the by-now familiar action (33) leads to the following equations of motion:

$$d\omega + \tfrac{1}{2}\varepsilon_{ab} e^a e^b = 0 \tag{61}$$

$$de^a + \omega \varepsilon^a{}_c e^c = 0 \tag{62}$$

$$d_A B = 0. \tag{63}$$

Equation (62) says that the spin-connection ω is torsion-free, and using this equation to express it in terms of e, we find that (61) is the statement that the metric $g(e)$ determined by the zweibeins has constant negative scalar curvature $R = -1$, which establishes the sought-for relation between flat $SO(2, 1)$-bundles on a Riemann surface and its moduli space within the field theory provided by our BF system. In order to arrive at a phase space describing surfaces which are not too singular we could, of course, restrict our attention to those flat bundles which have Euler class $2g-2$.

Then this relation is most easily understood in terms of uniformization and the relation between flat bundles and representations of the fundamental group. In the following, however, we do not want to further elaborate on this well-established fact, but rather explain how the above fits into Hitchin's deep and beautiful investigations of the self-duality equations in two dimensions. In particular, we shall see that the equations of motion (61) and (62) do in fact provide a solution to these equations, thus explaining from a more simple-minded point of view why self-duality equations can provide information on the structure of moduli space. At this point it is perhaps of interest to mention that there exists an alternative topological theory in two dimensions [31] which is based on the Abelian Higgs system. The moduli space of this theory is also closely related to the equations considered by Hitchin. His work, therefore, also provides a link between these two topological models.

Upon dimensional reduction of the four-dimensional self-duality equations $F = *F$ for the curvature of a principal $SO(3)$ bundle on R^4 to two dimensions these may be written in a conformally invariant way to make sense on an arbitrary Riemann surface M, thus giving rise to Hitchin's [19] *self-duality equations on a Riemann surface*,

$$F_A = -[\Phi, \Phi^*] \tag{64}$$

$$d''_A \Phi = 0. \tag{65}$$

Here the notation is the following: F_A is the curvature of a connection A on a principal $SO(3)$ bundle on M, $d''_A = d'' + A_{\bar{z}}\, d\bar{z}$ is the anti-holomorphic part of d_A with respect to a given complex structure on M, and $\Phi \in \Omega^{1,0}(M, \mathrm{ad}\, P^c)$ and $\Phi^* \in \Omega^{0,1}(M, \mathrm{ad}\, P^c)$ are complex combinations of the 3- and 4-components of the original four-dimensional connection.

In order to relate these equations to those of the gravity BF system we need —prior to making the appropriate identifications of the fields appearing in the two sets of equations—to understand what these have got to do with (a) flat $SO(2, 1)$-bundles and (b) constant negative curvature metrics. We shall deal with these two aspects in turn now, sketching how these relations can be established.

As far as (a) is concerned observe that Eqs. (64), (65) imply that the $PSL(2, C)$-connection $A + \Phi + \Phi^*$ is *flat*,

$$F_{A+\Phi+\Phi^*} = 0. \tag{66}$$

For irreducible flat $PSL(2, C)$-connections Donaldson [32] has proved that the converse is also true, and although this establishes the relation between the self-duality equations and flat bundles, this is not quite what we want yet, since our two-dimensional gravity theory is an $SO(2, 1)$-gauge theory, whereas (66) provides us with a flat $PSL(2, C) = SO(2, 1)^c$-connection. Fortunately, however, it can be shown that the part of the moduli space of solutions to (64), (65) can indeed be described by flat $SO(2, 1)$-connections. Even a summary of the proof of this fact lies beyond the scope of the present paper, so let us just mention that it relies on a conspiracy between the existence of a hyper-Kaehler metric on this moduli space and the involution $(A, \Phi) \to (A, -\Phi)$ (obviously mapping solutions into solutions) equipping it with a *real* structure.

Let us now turn to (b), i.e., the question of what the self-duality equations have to do with constant negative curvature metrics. Hitchin has constructed a family of solutions to these equations of the form $(A(g), \Phi(q))$. Here $A(g)$ is a connection on a certain rank two vector bundle $(K^{1/2} \oplus K^{-1/2})$, which is necessarily reducible to $U(1)$ and is the Levi–Civita connection corresponding to a metric g on M which by Eq. (64) has constant negative curvature for an appropriate choice of Φ. q is a holomorphic (because of (65)) quadratic differential, $q \in H^0(M, K^2)$, and Hitchin has shown how to construct from $(A, \Phi) \equiv (g, q)$ new metrics $g(q)$ with constant negative curvature [19, Theorem 11.2] such that any metric of the same constant negative curvature is isometric to one of these for some q. This provides then an explicit realization of the well-known isomorphisms $\mathscr{T} \approx H^0(M, K^2) \approx C^{3g-3}$, where $\mathscr{T}$ is Teichmüller space.

Having recalled all this, the relation between solutions to the self-duality equations (64), (65) and solutions to the two-dimensional topological gravity field equations (61), (62) is now fairly obvious. Since the connection-form $A = A_H$ of Hitchin is reducible to $U(1)$ we identify it—up to possibly a constant factor—with the spin-connection $J\omega$ of (58), whereas the fields $\Phi = \Phi(q)$ and $\Phi^*(q)$ are to be identified with the dz (resp. $d\bar{z}$) components of some zweibeins $e(q)$ corresponding to the metric $g(q)$.

Indeed, writing the BF $SO(2, 1)$-connection A_{BF} as

$$A_{BF} = J\omega + P_a e^a(q) = A_H + P_a e^a{}_z \, dz + P_a e^a{}_{\bar{z}} \, d\bar{z} = A_H + \Phi(q) + \Phi^*(q), \tag{67}$$

which we know is flat by (66) as it should be, we find—using the commutation relations (59)—that

$$[\Phi(q), \Phi^*(q)] = J g_{z\bar{z}}(q) \, dz \, d\bar{z}, \tag{68}$$

which implies that the first of the self-duality equations (64) is equivalent to the field-equation (61). On the other hand, the holomorphicity condition (65) is then just the no-torsion equation (62).

3.4. *Quantization and Metric-Independence of BF Systems in Dimensions* $\geqslant 4$

We have so far discussed Abelian BF systems in any and non-Abelian BF systems in two dimensions. For these models as well as for the non-Abelian BF system in three dimensions the standard BRST-argument was sufficient to establish the metric-independence of their partition functions. Non-Abelian BF-systems in four and more dimensions are on-shell reducible theories (i.e., some symmetries become reducible upon using the equations of motion). Thus quantization of these models requires the use of the full-fledged Batalin–Vilkovisky procedure [16]. As we shall see below this makes it much harder to establish their topological nature in general.

The reducibility is completely manifest in the B-field. In four dimensions, for example, the classical gauge-invariance (34), (35) of the action is

$$\begin{aligned} \delta A &= d_A \Lambda_0 \\ \delta B &= d_A \Lambda_1 + [B, \Lambda_0]. \end{aligned} \tag{69}$$

Consequently there is a zero-mode of these transformations when $\Lambda_1 = d_A \phi_0$ and the equation of motion $F_A = 0$ is used. This theory is easily quantized using the algorithm of [16] and (for a covariant gauge fixing on B) leads to the action

$$\begin{aligned} S_q = \int \operatorname{tr}(BF_A + \pi_1 d_A * B + \gamma d_A * \pi_1 + \bar{c}_1 d_A * d_A c_1 + \pi_0 d_A * c_1 \\ + \tau d_A * \bar{c}_1 + \bar{c}_0 d_A * d_A c_0 + c_0 d_A \bar{c}_1 d_A \bar{c}_1 + E_0 d * A + *\bar{\omega}\, \delta d_A \omega), \end{aligned} \tag{70}$$

where the non-zero BRST-transformations are

$$\begin{aligned} \delta B &= d_A c_1 - *[d_A \bar{c}_1, c_0] \\ \delta c_1 &= d_A c_0 \\ \delta \bar{c}_1 &= \pi_1 \\ \delta \bar{c}_0 &= \pi_0 \\ \delta \gamma &= \tau \\ \delta \bar{\omega} &= E. \end{aligned} \tag{71}$$

There are some things worth noting about the action (70). First, the ghost systems for A and B do not mix. This is true quite generally (i.e., in higher dimensions) provided that one uses covariant gauge conditions. Second, the gauge-fixing and the ghosts associated with B are exactly those found [33] for the Freedman–

Townsend model [34]. This is no accident since this model in an arbitrary number of dimensions is [35]

$$S_{\mathrm{FT}} = \int \mathrm{tr}(BF_A + A * A). \tag{72}$$

The only symmetry to gauge-fix here is the B-symmetry (due to the second term in (72)) and it corresponds to the B-symmetry in the BF systems, as the gauge-fixing of the B and A fields decouple. Yet another consequence of this split between the ghost systems is that the last two terms in (70) again do not introduce any metric dependence (by arguments which are by now quite familiar). We shall therefore in the following—where we discuss the quantization of non-Abelian BF systems in general—concentrate on the terms in the quantum action arising from the symmetries of B.

The ghost content is as usual given by the ghost triangle (which is the same in the Abelian and non-Abelian cases). The action S_q satisfying the master equation [16] may be expressed as an expansion in the dual fields,

$$S_q(\Phi, \Phi^*) = S(B) + \sum_n \Phi_n^* G_n(\Phi) + \sum_{n,m} \Phi_n^* \Phi_m^* G_{mn}(\Phi) + \cdots, \tag{73}$$

where Φ represents B and all the ghosts for ghosts and Φ^* represents the corresponding dual fields. This form of S holds also for a non-minimal set of fields.

We are now to some extent in a position to discuss the metric dependence of these theories. Unlike in the Abelian models, the gauge fixing terms here are not necessarily expressible as $\{Q, V\}$ for some V. The Batalin–Vilkovisky procedure guarantees only that $\{Q, S_q\} = 0$ (off-shell) and $\{Q, Q\} = 0$ (on-shell) and that the degrees of freedom are incorporated correctly. The Φ^*-fields are given in terms of the gauge fermion Ψ by $\Phi^* = \partial\Psi/\partial\Phi$ and the partition function is independent of the choice of Ψ. The variation of the partition function with respect to the metric is

$$\begin{aligned}\frac{\delta Z}{\delta g_{\mu\nu}} &= \left\langle \frac{\delta S_q}{\delta g_{\mu\nu}} \right\rangle \\ &= \left\langle \sum_n \Phi_n^* \frac{\delta}{\delta g_{\mu\nu}} G_n + \sum_{n,m} \Phi_n^* \Phi_m^* \frac{\delta}{\delta g_{\mu\nu}} G_{nm} + \cdots \right\rangle.\end{aligned} \tag{74}$$

The terms $\delta\Phi^*/\delta g_{\mu\nu}$ do not enter as they represent a change of Ψ which is an invariance of the theory. In the case of Abelian theories, only G_n is non-zero and $\delta G_n/\delta g_{\mu\nu} = 0$ as G_n represents the gauge transformation of the "nth field." In general,

$$\delta\Phi_n = \left.\frac{\delta S_q}{\delta\Phi_n^*}\right|_{\Phi^* = \partial\Psi/\partial\Phi} \tag{75}$$

and in the Abelian models the transformations are always in terms of the exterior derivative (there is no explicit metric). Our use of Ψ invariance here is precisely the statement in the Abelian models that a change in the metric induces a term $\langle\{Q, \delta V\}\rangle$ (V is the gauge fermion in that case).

Having used an equivalent amount of information as in the Abelian theory, we have not been able to establish the metric-independence of the partition function, so some more work and information is required here. It could, of course, have turned out that $\delta G_{mnp\cdots}/\delta g_{\mu\nu}=0$, but this is, for example, not realized in five dimensions, where G_{mn} is metric-dependent. One must therefore hope that the weaker condition

$$\left\langle \Phi_m^* \Phi_n^* \Phi_p^* \cdots \frac{\delta}{\delta g_{\mu\nu}} G_{mnp\cdots} \right\rangle = 0 \tag{76}$$

holds. And, since there are a number of indications and pieces of circumstantial evidence which point towards the metric independence of Z, in general, we are led to formulate the following

Conjecture[1]. The partition function of a non-Abelian BF system is independent of the metric in any dimension.

We have already seen that this is true in two and three dimensions, and below we shall give a proof of this in the four-dimensional case, using an argument which is quite likely to be applicable in higher dimensions as well. Before doing this though we want to present two further arguments in favour of the above conjecture.

First, on an arbitrary manifold in a background field gauge the non-Abelian theory at one loop level matches the Abelian system in the presence of a background (this corresponds to the large k limit discussed by Witten in [3]). Thus the one-loop result of the non-Abelian model is some power of the appropriate Ray–Singer torsion and is certainly metric-independent.

Second, we can prove metric independence of the partition function in any dimension n, provided that the n-manifold M is of the form $M=\Sigma_{n-1}\times R$: On such a manifold one may choose the "temporal" gauge for the A and B fields and, also, that gauge for all the consequent ghosts which need to be gauge-fixed. In this gauge all the interactions disappear and the theory devolves into several copies of an Abelian model (also in this gauge). As the partition function of the Abelian BF system is independent of the metric, so too is the non-Abelian partition function in this instance.

Returning now to the four-dimensional model (70) we find that the above conjecture is indeed realized. The form of S_q used to arrive at (70) is—writing it as in (73)—

$$S_q(\Phi, \Phi^*) = S(B) + \int B^* * d_A c_1 + B^* B^* c_0 + \Lambda^* \delta\Lambda, \tag{77}$$

[1] We have now been able to prove this conjecture, cf., the "Note Added in Proof" at the end of this paper.

where Λ^* represents all the dual fields except for B^* and $\delta\Lambda$ is the BRST-transformation for the Λ field. The gauge fermion is chosen to be

$$\Psi = \int B * d_A \bar{c}_1 + \cdots, \tag{78}$$

so that $B^* = * d_A \bar{c}_1 + \cdots$. The possible source of metric dependence is then contained in the term $B^* B^* c_0$, and although this term as a whole is metric-independent, we have to be more careful as we have already redefined the gauge fermion in such a way that any contribution $\delta B^*/\delta g_{\mu\nu}$ is zero. Fortunately, however, this offending term is indeed zero in the path-integral. This phenomenon has also been discovered (in a non-covariant gauge) by deAlwis *et al.* [33] in the Freedman–Townsend model, and the reason for this is the following: It is possible to assign a $U(1)$-number (charge) to the ghosts separately which is not the ghost number. For example, the ghosts c_0 and $\bar{c}_0$ (with ghost numbers 2 and -2) may be given the charges $(1, -1)$ with all other fields of charge zero. Then the interaction term is the only one with non-zero charge. It therefore does not contribute to the partition function which is then metric-independent.

4. Linking and Intersection Numbers and Observables

4.1. *Path Integral Representation of Linking and Intersection Numbers in Any Dimension*

So far we have restricted our attention exclusively to the partition function of BF systems. However, in topological field theories, observables other than the partition function play a very important role: they provide us with path-integral representations for smooth or topological invariants, as has, for instance, been shown by Witten [1] for the Donaldson polynomials [13] on the moduli spaces of instantons and [3] for the Jones polynomials [14] of knot theory. It also seems quite likely that the Mumford classes [36] of the moduli spaces of Riemann surfaces will arise as vacuum expectation values of the observables presented in [30]. It may, therefore, be expected that a careful treatment of the zero-modes arising in non-Abelian BF systems will permit the construction of observables on the moduli spaces of flat bundles which are known to have a rich geometrical and topological structure.

While this is certainly something that requires closer investigation we want to show in this section that interesting gauge invariant and metric independent observables may be defined in Abelian BF systems even if there are no zero modes present at all (as we shall assume). Indeed, we shall see below that these observables describe linking and intersection numbers of manifolds of any dimension. These comparably classical invariants are by their very definition metric-independent and are also homotopy-invariants in the sense that homotopic embeddings will give rise to the same intersection and linking numbers.

A clue as to why these should arise as observables in BF systems may be found in the work done recently on three-dimensional Chern–Simons theories. Witten [3] had advocated that in these theories Wilson loops are appropriate metric-independent and gauge-invariant objects. And although for non-Abelian gauge groups these are intimately related to the Jones polynomials of knot theory, which are unlikely to have any simple and straightforward generalization to other dimensions, the Abelian case looks more promising.

Another reason for restricting our attention to the Abelian case is the fact that in the non-Abelian case the Wilson loops (or surfaces) of the B-field are not invariant under the p-form symmetry (35). This problem does, of course, not appear in two dimensions, but the task of defining non-trivial observables for the non-Abelian BF systems in more than two dimensions remains.

Polyakov [21] has related the vacuum expectation values of Wilson loops in the Abelian Chern–Simons theory to the classical Gauss linking number of two loops. More precisely he has related the expectation value of a single Wilson loop to the somewhat singular concept of the self-linking number of a loop, but the generalization is obvious. However, even in the case of two loops there arises [3] a singular term in the computation of their linking number. As we shall see below the fact that in our theories we have two fields (A and B), instead of just one, allows us to bypass this difficulty altogether without the need for regularizing [21] or framing of the loops [3].

Reinterpreting [37, 38] this linking number as the intersection number of one loop with a disc bounded by the other loop, this "observable" has a natural generalization to higher dimensions. Indeed in the Abelian theories we have considered it is possible to define the appropriate analogues of Wilson loops.

If $\partial\Sigma$ and $\partial\Sigma'$ are disjoint compact and oriented p- and $(n-p-1)$-dimensional boundaries of two oriented submanifolds of an n-dimensional oriented manifold M, the fields B_p and A_{n-p-1} appearing in the action (1) of an Abelian BF system allow us to form the following metric independent and gauge invariant expressions ("Wilson surfaces"):

$$W_B(\Sigma) = \exp \int_{\partial\Sigma} B, \qquad W_A(\Sigma') = \exp \int_{\partial\Sigma'} A. \tag{79}$$

Since it is easily seen (either by assigning opposite "charges" to A and B or by the simple rules of Gaussian integration) that the expectation value of either one of these is one, the simplest object to consider is the vacuum expectation value of the product of them which we shall denote by

$$W(\Sigma, \Sigma') := \langle W_B(\Sigma)\, W_A(\Sigma') \rangle, \tag{80}$$

where the expectation value is, of course, taken with respect to the action $S(n, p)$ (1) of the Abelian BF system. In addition to the fact that this is clearly a metric independent and gauge invariant expression, the following two crucial properties are worth noting:

(1) $\dim(\partial\Sigma)+\dim(\partial\Sigma')=\dim(M)-1$

(2) the fundamental classes of $\partial\Sigma$ and $\partial\Sigma'$ are (obviously) homologous to zero in $H_*(M)$.

These are precisely [37] the conditions under which one expects to be able to define the *linking number of* $\partial\Sigma$ *and* $\partial\Sigma'$—as the intersection number of Σ and $\partial\Sigma'$ or $\partial\Sigma$ and Σ' determined in the following way [38]:

As the dimension of (say) Σ is equal to the codimension of $\partial\Sigma'$, they will generically intersect transversally at isolated points x_i. Having chosen orientations on M, Σ, and $\partial\Sigma'$ one assigns to each x_i the number $\text{sign}(x_i)$ which is $+1$ or -1 depending on whether the orientation of $(\Sigma, \partial\Sigma')$ at x_i coincides with that of M or not. The *intersection number of* Σ *and* $\partial\Sigma'$ is then defined as

$$\text{INT}(\Sigma, \partial\Sigma')=\sum_i \text{sign}(x_i) \tag{81}$$

which we also identify with the linking number $\text{LINK}(\partial\Sigma, \partial\Sigma')$. The reason why we have gone through all this is that we can show that the expectation value of the observable defined in (80) yields precisely the linking number defined above.

THEOREM. *Under the assumptions on* Σ *and* Σ' *stated above we have*

$$\log W(\Sigma, \Sigma')=\text{LINK}(\partial\Sigma, \partial\Sigma'). \tag{82}$$

Before turning to the proof of this theorem we shall—in order to acquire some familiarity with (82) and to understand why it is correct—look at some low-dimensional examples.

EXAMPLE 1. In $M=R^2$ we can, for instance, ask for the linking number of a point P and a circle γ. From the discussion above we know that we can define this in two ways: either as the intersection number of P with a disc D bounded by γ; or as the intersection number of γ with any ray ρ_P starting at P. In both cases a look at the definition (81) reveals that the result of our computation should be (if the theorem is correct) ± 1 or 0 depending on whether P lies inside or outside γ.

Choosing B to be a zero-form and A to be a one-form the observable we should be looking at is, therefore, (79), (80),

$$W_B(\rho_P)\, W_A(D)=\exp\left(B(P)+\oint_\gamma A\right). \tag{83}$$

By introducing "sources" for the A and B fields we can write the expectation value of (83) as

$$\left\langle \exp \int_M (B(x)\, J_2(x, P)+A(x)\, K_1(x, y))\right\rangle, \tag{84}$$

where y is a coordinate on γ and J_2 and K_1 are deRham currents [39] (dual to the embedding of P and γ into M) restricting the domain of integration to P and γ, respectively. As usual, gauge-invariance or consistency with the "equations of motion" following from the "action"

$$S(J, K) = \int (B\,dA + BJ + AK)$$

requires the currents J and K to be closed (this condition is of course empty for in this case for J—being a distribution-valued two-form in two dimensions).

The Gaussian integral over A and B in (84) is easily performed using the equations

$$\begin{aligned} A &= -\Delta^{-1}\,\delta J \\ B &= \Delta^{-1}\,\delta K, \end{aligned} \tag{85}$$

following from $dA + J = 0 = dB - K$, and leads to

$$\log W(\rho_P, D) = \int_M J(x, P)\,\Delta_x^{-1}\,\delta_x K(x, y). \tag{86}$$

Recalling that the whole purpose of J is to restrict the integrand to P and using Stoke's theorem (which is permitted here because of the compact support of K) to reexpress this as an integral over ρ_P, we find

$$\log W(\rho_P, D) = \int_{\rho_P} K(x, y). \tag{87}$$

Thus clearly the only contributions to this integral will come from those points where the ray ρ_P and the circle γ intersect. Now if P lies inside the circle, any ray starting at P will hit the circle exactly once and therefore (87) will give ± 1 in this case (depending on the chosen orientations). On the other hand, if P lies outside γ, the ray will either not intersect the circle at all or twice (with opposite relative orientation). In either case, the integral (87) will then be zero, confirming that the expectation of the observable (83) indeed computes the linking number of a point and a circle in this case.

Alternatively—by an integration by parts in (86)—we could have expressed the result as an integral over the disc D given (say) the orientation induced by $M = R^2$,

$$\log W(\rho_P, D) = \int_D J(P, x) = \int_D d^2x\,\delta(x - P), \tag{88}$$

which is clearly equal to 1, if P lies on the disc, and zero, otherwise, in agreement with the previous calculation.

Physically, of course, the above computations just reproduce Gauss's law that the total flux emmanating from a charge at P through an n-sphere is non-zero iff the charge is inside the n-sphere. This example therefore also straightforwardly generalizes to higher dimensions, the $U(1)$-gauge field A being replaced by an appropriate higher rank Abelian "Kalb–Ramond" field.

EXAMPLE 2. In three dimensions we have the by-now familiar [37, 38, 21, 3] example of the Gauss linking number of two loops γ and γ',

$$\mathrm{LINK}(\gamma, \gamma') = \frac{1}{4\pi} \oint_{\gamma} dx^i \oint_{\gamma'} dy^j \varepsilon_{ijk} \frac{(x-y)^k}{|x-y|^3} \tag{89}$$

which possesses as its electro-magnetic analogue Faraday's law. In the Abelian BF system $S(3, 1)$ this linking number arises as the expectation value

$$W(D, D') = \int \mathscr{D}A\, \mathscr{D}B \exp\left(\oint_{\gamma} B + \oint_{\gamma'} A\right) \tag{90}$$

and it may be checked that the offending self-linking number terms appearing in the analogous calculation for the Abelian Chern–Simons theory in three dimensions $(A = B)$ do not arise here.

After having discussed these examples we shall now turn to the proof of the general formula (82). The reason for having performed the computations in Example 1 in such an abstract way is that they carry over almost verbatim to any dimension.

Proof of the Theorem. As in Example 1 we introduce deRham currents $J = J_{\partial\Sigma}$ and $K = K_{\partial\Sigma'}$ for B and A to write

$$\begin{aligned} \log W(\Sigma, \Sigma') &= \log \left\langle \exp\left(\int_{\partial\Sigma} B + \int_{\partial\Sigma'} A\right)\right\rangle \\ &= \log \left\langle \exp \int_M (JB + KA)\right\rangle. \end{aligned} \tag{91}$$

Using (85) which is (modulo signs) valid in this case as well to perform the Gaussian integrals over A and B, one arrives at

$$\log W(\Sigma, \Sigma') = \int_M dx (J(x, z)\, \Delta_x^{-1}\, \delta_x K(x, y)),$$

where x, y, z are coordinates on M, Σ', and Σ, respectively, which reduces upon elimination of J or K and use of Stoke's theorem to either of the following two expressions:

$$\log W(\Sigma, \Sigma') = \int_{\Sigma} K_{\Sigma'} \tag{92}$$

$$= (-1)^{\dim\Sigma'(\dim\Sigma - 1)} \int_{\Sigma'} J_{\Sigma}. \tag{93}$$

Clearly—as above—(93) and (94) will only receive contributions (± 1) from points where Σ and $\partial\Sigma'$ or $\partial\Sigma$ and Σ' intersect, the correct orientation dependent sign arising as a consequence of the functional properties of the δ-functions in J and K. This completes the proof.

5. Conclusions

In this paper we have introduced a new class of topological field theories, and there are of course many questions that we have not dealt with fully yet and undoubtedly many more points that we have not even touched upon. We nevertheless hope to have conveyed to the reader a flavour of these models by exhibiting by means of some examples the rich mathematical structure they display despite their apparent simplicity.

One of the topics that has been left incomplete is the technical question of metric independence of non-Abelian BF systems in dimensions greater than four. In Section 3.4 we were able to show this for manifolds on which the global choice of an axial gauge is possible as well as for arbitrary manifolds for one loop. The gap still left between these two partial results certainly needs to be closed (cf. the "Note Added in Proof" at the end of this paper).

Another important question that has been left open is, what information one can extract from the observables that are available in the two-dimensional topological gravity model of Section 3.3. And whereas we have seen in Section 4.1 that there exists a large class of interesting observables in the Abelian BF systems, in the non-Abelian case one is confronted with the problem of finding any non-trivial observables.

In Section 2 we saw that the path-integral encodes a great deal of information about determinants—enough to prove metric-independence of the Ray–Singer torsion and its triviality in even dimensions, for instance. And here, at least, the prospects are favourable for making use of the path-integral to (re-)derive interesting results. Is it possible to derive Theorem 2.5 of Ray and Singer in this way?

And finally, recalling the Nicolai maps of Section 3: Are there analogous maps that trivialize the BF system in dimensions other than two? We have already seen that at the one loop level the BF systems and the Abelian Ray–Singer torsion models of Schwarz agree, so that in even dimensions it is in principle possible to trivialize the action while in odd dimensions the situation will be more complex.

Appendix

While a proof that regularized path integrals (including properties of the path measure) retain their naive properties is beyond the scope of this paper, it is at least possible to show that the regularization procedures used by Ray–Singer and Schwarz are compatible with BRST-invariance. To see what this statement amounts to consider the Chern–Simons theory in three dimensions. Its unregularized partition function is

$$Z = \int \mathscr{D}A \, \mathscr{D}E \, \mathscr{D}c \, \mathscr{D}\bar{c} \exp \int A \, dA + Ed * A + * \bar{c} \, \Delta c. \tag{1}$$

The last two terms come from $\{Q, \bar{c}d * A\}$, where Q is the ordinary Abelian BRST-operator. To introduce regulators into the gauge fixing terms in a BRST-invariant way (without formally altering Z), we adopt

$$\{Q, \bar{c}e^{G}d * A\} = Ee^{G}d * A + * \bar{c} \, \Delta e^{G} c, \tag{2}$$

where G is a function of the Laplacian such that it regularizes the determinants of the Laplacians involved. For example, the ghost term in (2) yields a determinant

$$\log \det \Delta e^{G} = \operatorname{Tr} G + \operatorname{Tr} \log \Delta \tag{3}$$

$$= \operatorname{Tr} \int_{\varepsilon}^{\infty} \frac{dt}{t} e^{-t\Delta} \tag{4}$$

or

$$= \frac{1}{\Gamma(s)} \operatorname{Tr} \int_{0}^{\infty} dt t^{s-1} e^{-t\Delta} \tag{5}$$

with the choices

$$G = -\int_{0}^{\varepsilon} \frac{dt}{t} e^{-t\Delta} \tag{6}$$

or

$$G = \log(\Delta(\Delta^{s} - 1)), \tag{7}$$

respectively. A completely regularized form of (1) would then be

$$Z_R = \int \mathscr{D}A \, \mathscr{D}E \, \mathscr{D}\bar{c} \, \mathscr{D}c \exp \int A \, de^{G/2} A + Ee^{G/2} d * A + * \bar{c} e^{G} \, \Delta c. \tag{8}$$

Notice that in the first two terms $e^{G/2}$ rather then e^{G} appears in order to regularize the first-order operators that appear there. As a consequence the second term does not match that in (2) and as the next step we have to relate Z and Z_R. This is achieved by scaling $A \to e^{-G/4} A$, $E \to e^{3G/4} E$, so that

$$Z_R = Z \det{}^{-1/4} e^{G(\Delta_0)} \det{}^{3/4} e^{G(\Delta_1)}, \tag{9}$$

where Z has the ghost structure of (2). But the right-hand side of (9) is of the correct form if Z has the form given in (1), and it is this that is guaranteed by the BRST-invariance. Roughly speaking, the BRST-invariance allows one to deal with regularization at the level of resolvents.

ACKNOWLEDGMENTS

G. T. thanks the members of the LPTHE, Paris, for their warm hospitality, and the CNRS for financial support. M. B. is grateful to the Italian Ministero della Pubblica Istruzione for a travel grant which allowed this work to be finally completed.

Note added in Proof. As we were completing this manuscript we received a preprint by Gary T. Horowitz (Exactly soluble diffeomorphism invariant theories, Santa Barbara Preprint ITP-NSF-88-178; *Commun. Math. Phys.* **125** (1989), 417) dealing with the same general class of theories as we do. The examples and techniques used are, however, substantially different, since the analysis of Horowitz relies on the canonical formalism, whereas ours is almost exclusively covariant. Consequently, there exists only an overlap with Sections 2.1 and 3.1 of our paper. While preparing the revised versions of this paper some works based on or directly related to it have appeared. G. T. Horowitz and M. Srednicki have also discovered the relation (Section 4) to linking and intersection numbers (Santa Barbara Preprint UCSB-TH-89-14). J. C. Wallet (Orsay Preprint IPNO-TH-8973; *Phys. Lett. B* **235** (1990), 71) has completed the quantization of the non-Abelian BF systems. Based on his results, we have now been able to establish the metric independence of the partition functions of non-Abelian BF systems in arbitrary dimensions (Marseille Preprint CPT-90/p. 2430). The reader may wish to keep this in mind while reading Sections 3.4 and 5 of this paper. And we have answered the question (raised in Section 5) in the affirmative, if there are Nicolai maps in more than two dimensions trivializing the BF systems (*Phys. Lett. B* **228** (1989), 64).

REFERENCES

1. E. WITTEN, *Commun. Math. Phys.* **117** (1988), 353,
2. E. WITTEN, *Commun. Math. Phys.* **118** (1988), 411.
3. E. WITTEN, *Commun. Math. Phys.* **121** (1989), 351.
4. L. BAULIEU AND I. M. SINGER, *Nucl. Phys. B (Proc. Suppl.)* **5** (1988), 12; R. BROOKS, D. MONTANO, AND J. SONNENSCHEIN, *Phys. Lett. B* **214** (1988), 91.
5. J. M. F. LABASTIDA AND M. PERNICI, *Phys. Lett. B* **212** (1988), 56.
6. D. BIRMINGHAM, M. RAKOWSKI, AND G. THOMPSON, *Phys. Lett. B* **214** (1988), 381.
7. D. BIRMINGHAM, M. RAKOWSKI, AND G. THOMPSON, *Nucl. Phys. B* **315** (1989), 577.
8. H. NICOLAI, *Phys. Lett. B* **89** (1980), 341; **117** (1982), 408.
9. S. CECOTTI AND L. GIRARDELLO, *Phys. Lett. B* **110** (1982), 39.
10. M. F. ATIYAH, *Proc. Symp. Pure Math.* **48** (1988), 285.
11. M. GROMOV, *Invent. Math.* **82** (1985), 307.
12. A. FLOER, *Commun. Math. Phys.* **118** (1988), 215.
13. S. K. DONALDSON, *Topology* **29** (1990), 257.
14. V. F. R. JONES, *Bull. Amer. Math. Soc.* (*N.S.*) **12** (1985), 103.
15. E. WITTEN, *Nucl. Phys. B* **311** (1988/89), 46.
16. I. A. BATALIN AND G. A. VILKOVISKY, *Phys. Rev. D* **28** (1983), 2567.
17. A. S. SCHWARZ, *Commun. Math. Phys.* **67** (1979), 1.
18. D. B. RAY AND I. M. SINGER, *Adv. Math.* **7** (1971), 145.
19. N. J. HITCHIN, *Proc. London Math. Soc. (3)* **55** (1987), 59; "Gauge Theories on Riemann Surfaces," Lectures given at the ICTP College on Riemann Surfaces, December 1987.

20. S. G. Rajeev, *Phys. Lett. B* **212** (1988), 203.
21. A. M. Polyakov, *Mod. Phys. Lett. A* **3** (1988), 325.
22. P. K. Townsend, *Phys. Lett. B* **88** (1979), 97; M. R. Namazie and D. Storey, *J. Phys. A: Math. Gen.* **13** (1980), 161; W. Siegel, *Phys. Lett. B* **93** (1980), 170.
23. R. Delbourgo and P. D. Jarvis, *J. Phys. A: Math. Gen.* **16** (1983), L275.
24. A. S. Schwarz and Yu. S. Tyupkin, *Nucl. Phys. B* **242** (1984), 436.
25. A. M. Polyakov, *Nucl. Phys. B* **120** (1977), 429.
26. D. Amati and A. Rouet, *Nuovo Cimento A* **50** (1979), 265.
27. O. Babelon, *Nucl. Phys. B* **131** (1977), 519.
28. G. 't Hooft, *Nucl. Phys. B* **75** (1974), 461.
29. J. M. F. Labastida, M. Pernici, and E. Witten, *Nucl. Phys. B* **310** (1988), 611.
30. D. Montano and J. Sonnenschein, *Nucl. Phys. B* **324** (1989), 348.
31. F. A. Schaposnik and G. Thompson, *Phys. Lett. B* **224** (1989), 379.
32. S. K. Donaldson, *Proc. London Math. Soc. (3)* **55** (1987), 127.
33. A. Diaz, *Phys. Lett. B* **203** (1988), 408; S. P. deAlwis, M. T. Grisaru, and L. Mezinescu, *Nucl. Phys. B* **303** (1988), 57.
34. D. Z. Freedman and P. K. Townsend, *Nucl. Phys. B* **177** (1981), 282.
35. A. A. Slavnov and S. A. Frolov, Padova Preprint DFDP 13/87.
36. D. Mumford, Towards an enumerative geometry of the moduli space of curves, *in* "Progress in Mathematics," Vol. 36, pp. 271–328, Birkhäuser, Boston, 1983.
37. R. Bott and L. W. Tu, "Differential Forms in Algebraic Topology," Springer-Verlag, New York, 1986.
38. B. Dubrovnic, S. Novikov, and A. Formenko, "Geometrie Contemporaire," Vol. 2, Nauka, Moscow, 1982.
39. G. deRham and K. Kodaira, "Harmonic Integrals," Princeton Univ. Press, Princeton, NJ, 1953.