A NEW CLASS OF TOPOLOGICAL FIELD THEORIES AND THE RAY-SINGER TORSION

Matthias BLAU
International School for Advanced Studies (SISSA), Strada Costiera 11, I-34014 Trieste, Italy

and

George THOMPSON
LPTHE, Université Paris VI, Tour 16, 1er étage, 4 Place Jussieu, F-75252 Paris Cedex 05, France

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Recently proposed topological theories with classical action $\mathrm{Tr}\, B\wedge F$ are investigated. Their partition function is shown to be an integral, over the moduli space of flat connections with measure determined by the Ray–Singer torsion.

1. Introduction

Topological field theories have aroused some interest since last year when Witten in separate papers first showed that Donaldson's polynomials have a natural setting in field theory [1] and secondly that the three-dimensional Chern–Simons theory gives new insight into conformal field theories and to three-dimensional gravity [2,3].

A natural question that arises is if it is possible to generalize the Chern–Simons models to higher dimensions? This is indeed the case and has been done by Horowitz [4] and ourselves [5]. The idea here is to write down a metric independent action, which one can quantize in the path integral and which leads to a theory with no physical degrees of freedom, but is nonetheless interesting in that it has a finite dimensional phase space, which in turn corresponds to some moduli space.

In refs. [4,5] we are led to consider the classical theories with action

$$S=\mathrm{Tr}\int B\wedge F\,, \qquad (1)$$

where F is the curvature two-form of a connection A on a principle bundle P over M with (simple) structure group G. B is a section of Ω^{n-2}(M,adP) i.e. locally a Lie algebra valued form on M transforming under the adjoint representation of G. (1) represents a generalization of the models considered by Schwarz [6] namely

$$S=\mathrm{Tr}\int H\wedge \mathrm{d}_A C \qquad (2)$$

where d_A is the covariant derivative of a flat bundle and H and C are a p-form and $(n-p-1)$-form (with values in the bundle) respectively. These models (2) are intimately related [6] to the Ray–Singer torsion [7]. In ref. [5] we pointed out that at the one loop level the partition function of (1) becomes the partition function of (2) when H is chosen to be an $(n-2)$-form (otherwise they are related by a power).

The purpose of this short note is to show that the partition function of (1) is the sum (integral) over the moduli space of flat connections modulo gauge transformations of the partition function for a flat connection given by (2), i.e.

$$Z_{(1)}=\int_{\mathcal{M}} Z_{(2)}\,. \qquad (3)$$

Furthermore, in even dimensions, as the Ray–Singer torsion is trivial [7] (a field theory proof of

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this may be found in ref. [5]) this expression simplifies to

$$Z_{(1)} = \int_{\mathcal{M}} \mathbf{1} = V_{\mathcal{M}},$$

where $V_{\mathcal{M}}$ is the volume of the moduli space. In general $Z_{(2)}$, that is the Ray–Singer torsion, provides the measure on the moduli space. Eq. (3) requires further explanation especially with regard to questions of zero modes, this is best done after we have developed the formalism a little and in terms of the examples. Recall that as the Ray–Singer torsion is a topological invariant, $Z_{(2)}$ is metric independent, we will also come back to this in the text.

The equations of motion that arise from (1) are

$$F=0, \quad d_A B=0. \tag{4a,b}$$

One now expands the fields about the solutions to (4a, 4b)

$$A \Rightarrow A+Q, \quad B \Rightarrow B_c + B, \tag{5}$$

where now A and B_c are the classical solutions and A is taken to satisfy $\delta_A A=0$ so as to distinguish gauge inequivalent solutions (δ_A is the covariant coderivative). Firstly we want to show that B_c (the holonomy) does not enter. (1) is now expressed as

$$S = \mathrm{Tr} \int B_c \wedge Q \wedge Q + B \wedge d_A Q + B \wedge Q \wedge Q. \tag{6}$$

The path integral over B leads to a delta function

$$\delta(d_A Q + Q \wedge Q),$$

so that $\int B_c \wedge Q \wedge Q$ in (5) may be replaced with $\int B_c \wedge d_A Q$ which vanishes. Hence in the following we will simply set $B_c=0$. The action of interest throughout is then

$$S = \mathrm{Tr} \int B \wedge F(A+Q). \tag{7}$$

At this point we are in a position to discuss the question of the B field zero modes, that is, the B_c. We should in principle integrate (sum) over all the gauge inequivalent B_c but as there is no weighting for these in the action, one will obtain the volume of the moduli space of B zero modes, which is suppressed on the right-hand side of (3). On the other hand when taking expectation values of products of fields on the moduli space (of B and A) it must be remembered that there is still this integration to be performed.

2. Two-dimensional theory

In the case of two dimensions the field B entering (1) is a zero form and does not possess its own gauge transformation (though it transforms homogeneously under the A fields gauge transformation), so we need gauge fix only the vector potential A. We do this covariantly around the classical solutions,

$$\delta_A Q = 0.$$

The corresponding BRST invariant action is

$$S_2 = S + \mathrm{Tr} \int {*E} \wedge \delta_A Q + {*\omega'} \wedge \delta_A d_{A+Q} \omega. \tag{8}$$

This is the gauge fixed version of (1) in a background. In principle we must integrate (or sum) over all flat connections not related by gauge transformations, in the following this is implicitly assumed, except in some instances it will be made explicit.

To put (7) into the form (2) we change variables. Let

$$\xi = {*F}(A+Q), \quad \eta = \delta_A Q. \tag{9}$$

Then S_2 becomes

$$\mathrm{Tr} \int {*B} \wedge \xi + {*E} \wedge \eta + {*\omega'} \wedge \delta_A d_{A+Q} \omega, \tag{10}$$

however, we still need to take into account the jacobian of the change of variables

$$\mathrm{Jac} = \det[\delta Q / \delta(\xi, \eta)] = \det[\delta(\xi, \eta)/\delta Q]^{-1}$$
$$= \det[({*d}_{A+Q}, \delta_A)]^{-1}.$$

This has a simple path integral representation, namely

$$\mathrm{Jac} = \int dH\, dC\, d\sigma \exp\left(\mathrm{Tr} \int H \wedge d_{A+Q} C + {*\sigma} \wedge \delta_A C\right),$$

where all the fields are bosonic, C is a one-form while H and σ are zero-forms. The integrals over B and E constrain Q to be zero by the assumption implicit in the split made in (5). Our final action is therefore of the desired form,

$$S_2=\mathrm{Tr}\int H\wedge d_A C+ *\sigma\wedge \delta_A C+ *\omega'\wedge\delta_A d_A\omega. \quad (11)$$

(11) is the gauge fixed form of (2) where the gauge symmetry is

$$C'=C+d_A\lambda .$$

This establishes in this example the desired result (3). When the cohomology groups associated with d_A, $H^*(M, adP)$, are taken to be trivial as then (11) leads to the Ray–Singer torsion (the triviality of $H^*(M, adP)$ is part of the definition of the torsion). When the fields have zero modes these must also be taken into account, how to do this is explained in refs. [5,8]. So one more interpretational adjustment of (3) is that on the right-hand side one has the modified Ray–Singer torsion in the presence of zero modes.

In ref. [5] this model was used to describe a theory whose configuration space is the moduli space of Riemann surfaces, and is a variant of the theories proposed in ref. [9] and is closely related to the work of Hitchin [10]. Furthermore, this change of variables may be made in two-dimensional Yang–Mills theory [5] and leads to the results of Rajeev [11] which are based on a canonical analysis. This comes about as follows. In two dimensions the Yang–Mills action is obtained by taking the action (7) and augmenting it with $\mathrm{Tr}\int *B\wedge B$, the B integration giving rise to the usual F^2 action. The change of variables (9) is then still suited to trivializing the action. Further details may be found in ref. [5].

3. Three-dimensional theory

When considering the model in three dimensions the new feature that arises is that the B field also posseses a gauge symmetry above and beyond that of its transformation associated with the gauge transformation of the vector potential A. This symmetry is a consequence of the Bianchi identity, indeed we see that for any dimension $n\geqslant 3$,

$$\delta B=d_{A+Q}\Lambda . \quad (12)$$

is an invariance of (7), where Λ is an $(n-3)$-form. The BRST invariant gauge fixed action in a background field A is

$$S_3=\mathrm{Tr}\int S+ *E\wedge\delta_A Q+ *\Pi\wedge\delta_{A+Q}B$$
$$+ *\omega'\wedge\delta_A d_{A+Q}\omega+ *\chi'\wedge\delta_{A+Q}d_{A+Q}\chi ,$$

where the χ fields are the ghosts associated with the symmetry (12). The required change of variables is

$$\xi= *F(A+Q)+d_{A+Q}\Pi,\quad \eta=\delta_A Q \quad (13)$$

Notice that ξ differs from the choice made in two dimensions (9). The reason is that if one wants to obtain a gaussian action in the new variables then ξ must be the precise combination of fields that couple to B.

In terms of these fields S_3 is rewritten as

$$S_3=\mathrm{Tr}\int *B\wedge\xi+ *E\wedge\eta$$
$$+ *\omega'\wedge\delta_A d_{A+Q}\omega+ *\chi'\wedge\delta_{A+Q}d_{A+Q}\chi, \quad (14)$$

while the jacobian adds the following terms (following precisely the two-dimensional example):

$$\mathrm{Tr}\int H\wedge d_{A+Q}C+ *\sigma\wedge\delta_A C+ *\rho\wedge\delta_{A+Q}H , \quad (15)$$

where H and C are one-forms while σ and ρ are zero-forms. From (14) and (15) we have both Q and Π zero. On integrating out both B and E, and adding (14) and (15) we arrive at the action

$$S_3'=\mathrm{Tr}\int H\wedge d_A C+*\sigma\wedge\delta_A C+ *\omega'\wedge\delta_A d_A\omega$$
$$+ *\chi'\wedge\delta_A d_A\chi+ *\rho\wedge\delta_A H , \quad (16)$$

which once more is the appropriately quantized version of (2) in three dimensions, thus establishing (3) in this case.

Notice now that the map (13) has zeros not only at $\Pi=0$ but also at $d_A\,\Pi=0$ (with $Q=0$ in both cases), which can occur when $H^0(M, adP)$ is not trivial. This in turn is reflected in the zero mode of σ. Once more this implies a modification of the Ray–Singer torsion and of what is the argument of the integral in (3). For nontrivial $H^*(M,adP)$ this situation also persists in higher dimensions, but we will refrain from reiterating this in the next sections.

4. $n \geqslant 4$ dimensions

As already discussed in the introduction in higher dimensions the symmetry (12) leads to the usual phenomenon of ghosts for ghosts, a property shared by both theories (7) and (2). Indeed they have the same ghost content in the sense that they share common ghost kinetic terms as determined by their ghost triangles [8]. A real difficulty arises however in that the ghost structure of the models with action (7) when quantized in the Batalin–Vilkovisky framework [12], includes at least cubic ghost interactions and possibly higher order terms (as far as we know the explicit expressions in five dimensions and higher have not been worked out yet in the lagrangian formalism). A simple argument in four dimensions [5,13] shows that the cubic term that makes an appearance does not contribute to the partition function (in the absence of ghost zero modes). It is not clear that we will be so lucky in higher dimensions, however, we will at the end present arguments in favour of this conclusion. So for now we ignore this complication, and presume that the ghost terms are as those for (12), that is entering either with a kinetic operator $\delta_{A+Q}\mathrm{d}_{A+Q}$ or as multiplier fields enforcing a gauge condition.

To determine the required change of variables in n dimensions it is easiest to make use of the Batalin–Vilkovisky ghost triangle of fig. 1. Included in the diagram to the left of the edge of the triangle are various multiplier fields associated with the (anti or Kallosh–Nielsen) ghost that they lie directly to the left of, so for example the BRST variation of χ'_n is Π_n. The change of variables that we performed in the two and three dimensional examples were to define a new field that included all the terms that B was coupled to in the action (13) which included Π_1 (in the notation of fig. 1, we called it Π previously). However Π_l couples to χ'_2 as does Π_3 so we define a new field ξ_1 which is the precise combination of Π_1 and Π_3 that couples to χ'_2, Π_3 also couples to χ'_4 as does Π_5 so a new field ξ_2 is defined with this precise combination of Π_3 and Π_5 and so on, in this way we arrive at the field redefinitions

$$\xi = {}^*F(A+Q) + \mathrm{d}_{A+Q}\Pi_1\,, \quad \eta = \delta_A Q\,,$$

$$\xi_i = \delta_{A+Q}\Pi_{2i-1} + \mathrm{d}_{A+Q}\Pi_{2i} + 1\,, \quad 1 \leqslant i \leqslant n-2\,.$$

In n dimensions the Π_j for $j \geqslant n-3$ are zero. The jacobian of such a change of variables is not degenerate (if the forms took values in R rather than in a bundle the maps above would be the Hodge decomposition without harmonic pieces). The jacobian when expressed as a path integral adds to the action the terms,

$$\mathrm{Tr}\int \mathrm{H}\wedge \mathrm{d}_{A+Q}C + {}^*\sigma_1 \wedge \delta_A C + {}^*\rho_1 \wedge \delta_{A+Q}H$$

$$+ \sum {}^*\sigma_i \wedge (\delta_{A+Q}\rho_{2i-1} + \mathrm{d}_{A+Q}\rho_{2i+1})\,.$$

With the sum from $i=1$ to $i=n-3$ (recall that the ρ_i with $i \geqslant n-3$ vanish). Now the integration over E and χ'_{2i+1} will set Q and the ξ_i to zero. However, by simple relabeling $\sigma_i \Rightarrow \chi_i$ and $\rho_i \Rightarrow \Pi_i$ we arrive back to the original action, except that $B\wedge F$ is replaced with $H\wedge \mathrm{d}_A C$. With $Q=0$ the ghost terms of the $B\wedge F$ system devolve to those of $H\wedge \mathrm{d}_A C$. This establishes (3) in general providing one may ignore the complications of ghost interactions alluded to above.

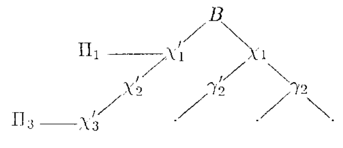

Fig. 1. Ghost triangle for the B-part of the general $B\wedge F$ system, with some of the multiplier fields displayed.

5. Ghost interactions and metric independence

Within the Batalin–Vilkovisky framework ghost interactions arise in the following way. On quantizing the B field the ghost action that arises is of the form $\chi'_1 \wedge \delta_{A+Q}\mathrm{d}_{A+Q}\chi_1$. This term does not have an off-shell invariance. However if one transforms χ_1 to $\chi_1 + \mathrm{d}_{A+Q}\chi_2$ the varied ghost term will be $\chi'_1 \wedge \delta_{A+Q}[F(A+Q)\wedge\chi_2]$. Now as B couples to $F(A+Q)$ we see that there is an on shell invariance and that a way of realizing this is to alter the B transformation to get rid of the unwanted term. In fact such terms $(\chi'_i \wedge \delta_{A+Q}[F(A+Q)\wedge\chi_{i+1}])$ will arise in all the variations of the ghost for ghost.

The implications of this are quite important. While one may doctor the B (and also ghost) transformation to soak up the unwanted terms, the gauge fixing pieces are not invariant under the new transformation rules. It is precisely for this reason that cubic (and possibly higher) ghost interactions are introduced in the Batalin–Vilkovisky procedure, their variation cancels that coming from the gauge fixing parts. As a consequence of the altered variations of the fields and the existence of the interaction terms one cannot write the complete set of gauge fixing and ghost terms as the BRST variation of some functional. This in turn means that we have no easy way of establishing the metric independence of the partition function, for recall that [1,5,14] if the metric dependence is contained in a BRST commutator and the BRST operator is metric independent then by BRST invariance the partition function is metric independent. We are faced with a situation in which the ghost terms cannot be expressed as a BRST commutator and because of the extra terms in the transformation rules the BRST operator is metric dependent.

The situation in four dimensions is heartening. There the ghost interaction term is not relevant in that it does not contribute to the partition function Z and the issue of metric dependence is clarified (there is none (see ref. [5]). There is an argument as to why this should continue to be the case in higher dimensions which we now present. Rather than follow the Batalin–Vilkovisky algorithm we adopt a rather different point of view. Let us decide not to alter the BRST charge (so that it does not pick up a metric dependence) then for example the variations of the fields B, χ_i are,

$$\delta B = \mathrm{d}_{A+Q}\chi_1 , \quad \delta\chi_i = \mathrm{d}_{A+Q}\chi_{i+1} . \tag{17}$$

Now the ghost terms $\chi_i' \wedge \delta_{A+Q}\mathrm{d}_{A+Q}\chi_i$ are not invariant under this change, however, because of the B integration there is a delta function constraint setting $F(A+Q)$ to zero. In the path integral the non invariant terms are then zero by this constraint. So relaxing the usual rules for deriving a BRST invariant action to allow one whose BRST variation is zero in the path integral brings us to the situation we have been considering throughout the previous section. Furthermore, $(Q_{\mathrm{BRST}})^2 \sim F$ which also vanishes "in" the path integral (this holds for Z and also in the presence of sources providing there is no source for B). Accepting this philosophy one is led directly to the metric independence of Z and (via our previous discussion) to the identity (3).

This argument as it stands is not complete of course but it is suggestive. Indeed the four-dimensional case where the ghost interactions do not contribute is an example of the type of quantization that we envisage. A more complete analysis of this procedure would require a better understanding of the geometric basis of the Batalin–Vilkovisky algorithm. In this direction one expects Schwarz's notion of a resolvent [6] will play an important role.

Note added

On completing this manuscript we received papers by Karlhede and Roček [15] and by Witten [16] which have some overlap with the present work.

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